"**Some Wave Properties of Helical Conductors**", J H Bryant,
Electrical Communication, Vol. 31(1), 1954.

Some Wave Properties of Helical Conductors

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THE wave properties of a helically conducting cylinder in free space have been studied in detail.[1] In delay lines using spiraled conductors as well as in some traveling-wave-tube applications, the wave properties of the helical structure are modified by the presence of additional coaxial conductors.

The factors of interest in traveling-wave-tube work are the velocity of propagation and the impedance function $E_z^2/\beta^2 P$ (see section 6 for glossary of symbols) relating the longitudinal field to the total power flowing. This latter field determines the degree of interaction of the wave and the moving electron stream.

The problem considered is illustrated in Figure 1, which shows the infinitely thin helically conductive cylinder of radius a and coaxial inner and outer uniformly conductive cylinders of radiuses c and b, respectively. Four conditions are possible depending on the presence, absence, or combination of conducting cylinders. These conditions are considered separately in the subsequent sections.

1. *Helix in Free Space*

The wave properties of a helical conductor in free space have been adequately covered in the literature. The results are recorded here for information and comparison with other conditions. For this condition, where $c = 0$ and $b = \infty$, the radial propagation constant γ is given by

$$(\gamma a)^2 \frac{I_0(\gamma a)K_0(\gamma a)}{I_1(\gamma a)K_1(\gamma a)} = (ka \cot \psi)^2. \tag{1}$$

The factor $F(\gamma a)$ in the impedance parameter, $(E_z^2/\beta^2 P)^{1/3} = (\beta/k)^{1/3}(\gamma/\beta)^{4/3}F(\gamma a)$, is given by

$$F(\gamma a) = \left[\frac{\gamma a}{240}\frac{I_0(\gamma a)}{K_0(\gamma a)}\right]\left[\left(\frac{I_1(\gamma a)}{I_0(\gamma a)} - \frac{I_0(\gamma a)}{I_1(\gamma a)}\right) + \left(\frac{K_0(\gamma a)}{K_1(\gamma a)} - \frac{K_1(\gamma a)}{K_0(\gamma a)}\right) + \frac{4}{\gamma a}\right]^{1/3}. \tag{2}$$

2. *Helical Conductor Inside a Coaxial Conductive Cylinder*

This condition is the one most likely to be met in practice. For this condition, $c = 0$ and b has a finite value. The radial propagation constant γ is given[2] by

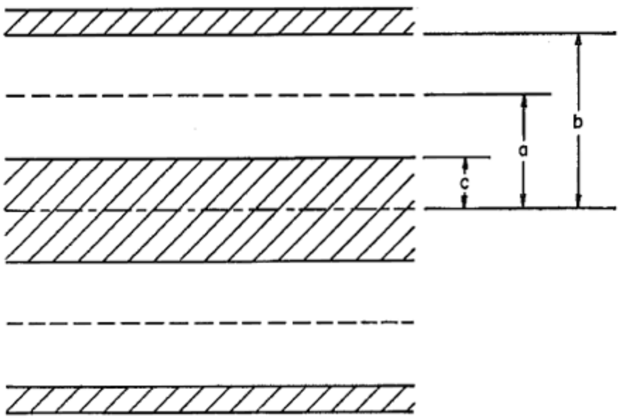


Figure 1—Helically conductive cylinder with inner and outer coaxial conductive cylinders.

$$\left(\frac{k}{\gamma}\cot\psi\right)^2 = \frac{I_0(\gamma a)K_0(\gamma a)}{I_1(\gamma a)K_1(\gamma a)}\left[\frac{1 - \dfrac{I_0(\gamma a)K_0(\gamma b)}{K_0(\gamma a)I_0(\gamma b)}}{1 - \dfrac{I_1(\gamma a)K_1(\gamma b)}{K_1(\gamma a)I_1(\gamma b)}}\right]. \tag{3}$$

The solution is shown plotted in Figure 2 with b/a as a parameter. As can be seen, the effect is to reduce the low-frequency dispersion.

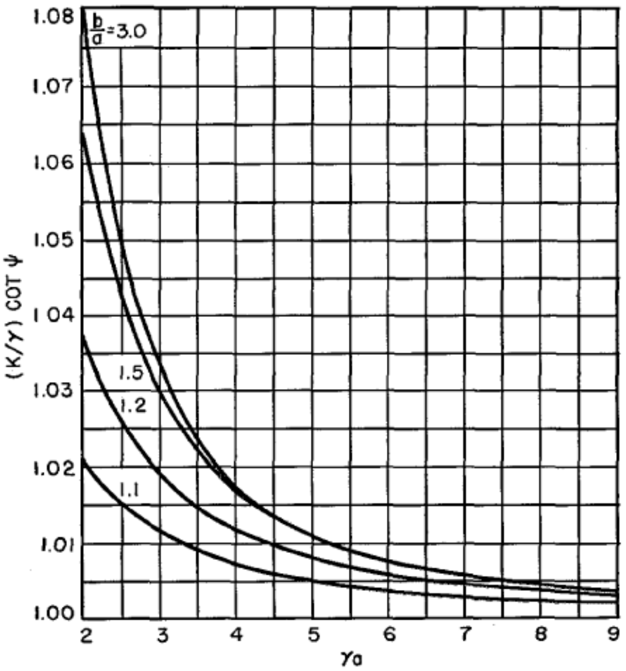
The factor $F(\gamma a, \gamma b)$ in the impedance parameter is given by (4).

[1] J. R. Pierce, "Traveling Wave Tubes," D. Van Nostrand Company, New York, New York; 1950.

[2] L. N. Loshakov and E. B. O'Derogge, "On the Theory of the Coaxial Spiral Line," *Radiotecknika* (Moscow), volume 3, pages 11–20; March/April, 1948.

$$F(\gamma a,\gamma b) = \left\{ \frac{(\gamma a)^2}{240} \left\{ \left[1 + \frac{I_0^2(\gamma a)}{\underline{R}I_1^2(\gamma a)} \right] \left[M(\gamma a) \right] + I_0^2(\gamma a) \left[\left(\underline{\frac{K_0^2}{H} + \frac{K_1^2}{JR}} \right) \left(\frac{b^2}{a^2} M(\gamma b) - M(\gamma a) \right) \right. \right. \right.$$

$$\left. \left. \left. + \left(\underline{\frac{I_0^2}{H} + \frac{I_1^2}{JR}} \right) \left(\frac{b^2}{a^2} N(\gamma b) - N(\gamma a) \right) + 2 \left(\underline{\frac{I_0 K_0}{H} - \frac{I_1 K_1}{JR}} \right) \left(\frac{b^2}{a^2} P(\gamma b) - P(\gamma a) \right) \right] \right\} \right\}^{-1/3}, \quad (4)$$



where I_0, I_1, and I_2 are Bessel functions of an imaginary argument of the first kind of order zero, one, and two, respectively; and K_0, K_1, and K_2 are Bessel functions of an imaginary argument of the second kind of order zero, one, and two, respectively.

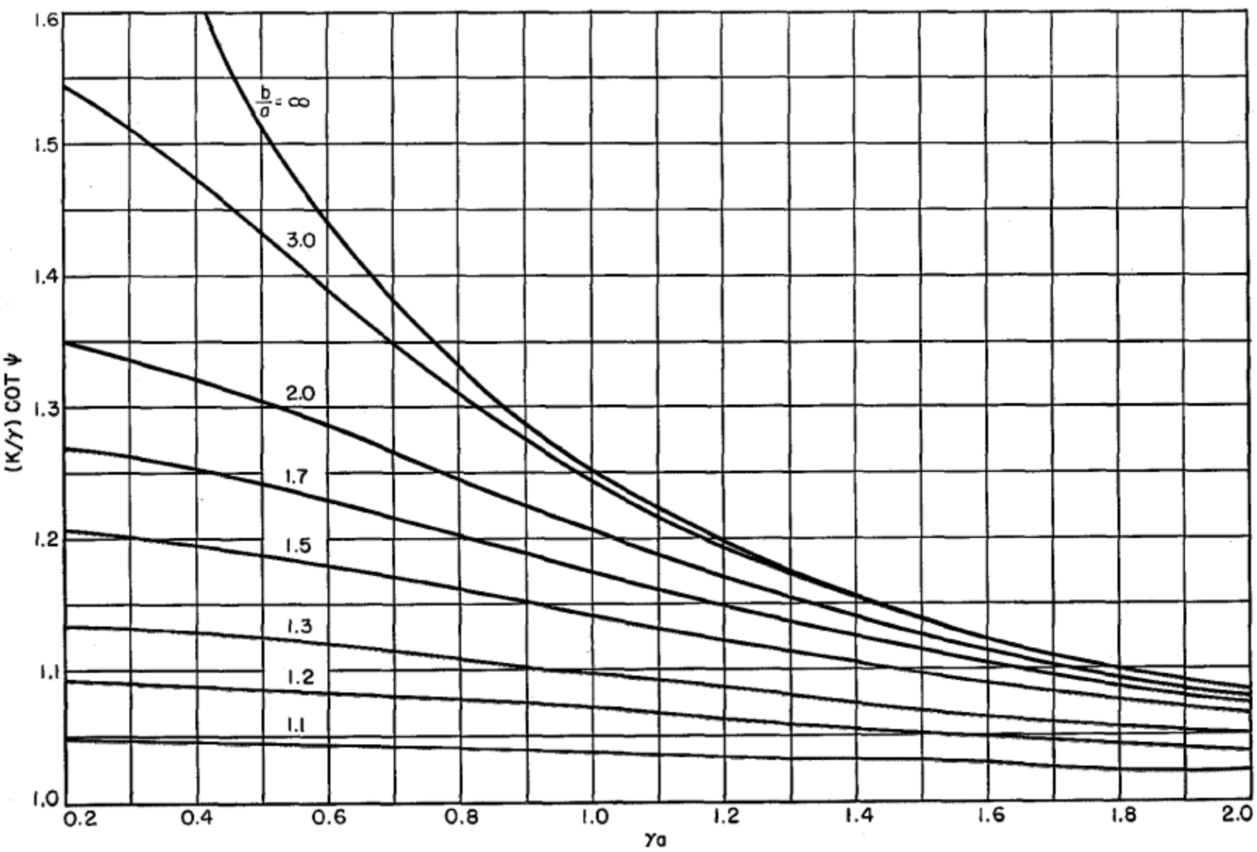
The broken-line underscoring denotes the argument γa for the functions and the solid-line underscoring denotes the argument γb.

The quantities H, J, R, $M(x)$, $N(x)$, and $P(x)$ are defined by the following equations.

$$H \equiv [I_0(\gamma a)K_0(\gamma b) - I_0(\gamma b)K_0(\gamma a)]^2$$

$$J \equiv [I_1(\gamma a)K_1(\gamma b) - I_1(\gamma b)K_1(\gamma a)]^2$$

Figure 2—Below and at left, (k/γ) cot ψ, a quantity proportional to velocity, plotted as a function of γa, a quantity proportional to frequency, for a helically conductive cylinder within a coaxial conductive cylinder.



$$R \equiv \frac{I_0(\gamma a)K_0(\gamma a)}{I_1(\gamma a)K_1(\gamma a)} \left[\frac{1 - \dfrac{I_0(\gamma a)K_0(\gamma b)}{K_0(\gamma a)I_0(\gamma b)}}{1 - \dfrac{I_1(\gamma a)K_1(\gamma b)}{K_1(\gamma a)I_1(\gamma b)}} \right]$$

$$M(x) \equiv I_1^2(x) - I_0(x)I_2(x)$$

$$N(x) \equiv K_1^2(x) - K_0(x)K_2(x)$$

$$P(x) \equiv I_1(x)K_1(x) + I_2(x)K_2(x).$$

Figure 3 shows $F(\gamma a, \gamma b)$ as a function of γa for various radiuses of the surrounding cylinder. The effect of the surrounding cylinder has been to reduce the impedance parameter as b/a is decreased. The physical interpretation of this is that as b/a is reduced, more of the radio-frequency energy flows between the helix and the cylinder, thus reducing E_z^2 on the axis.

The derivations of (3) and (4) are given in the appendix.

3. Helical Conductor Surrounding a Coaxial Conductive Cylinder

For this condition, radius c is finite and less than a, while b is infinite. The radial propagation

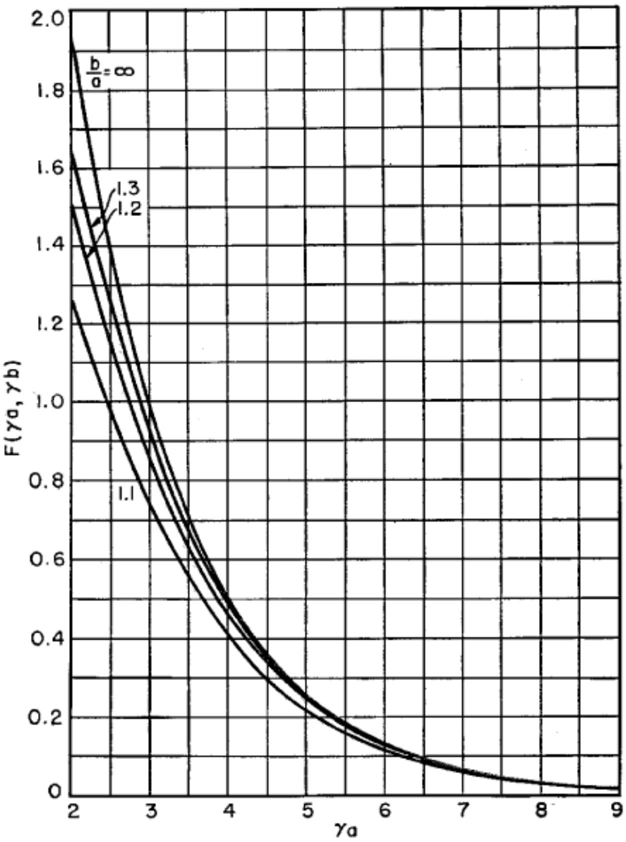
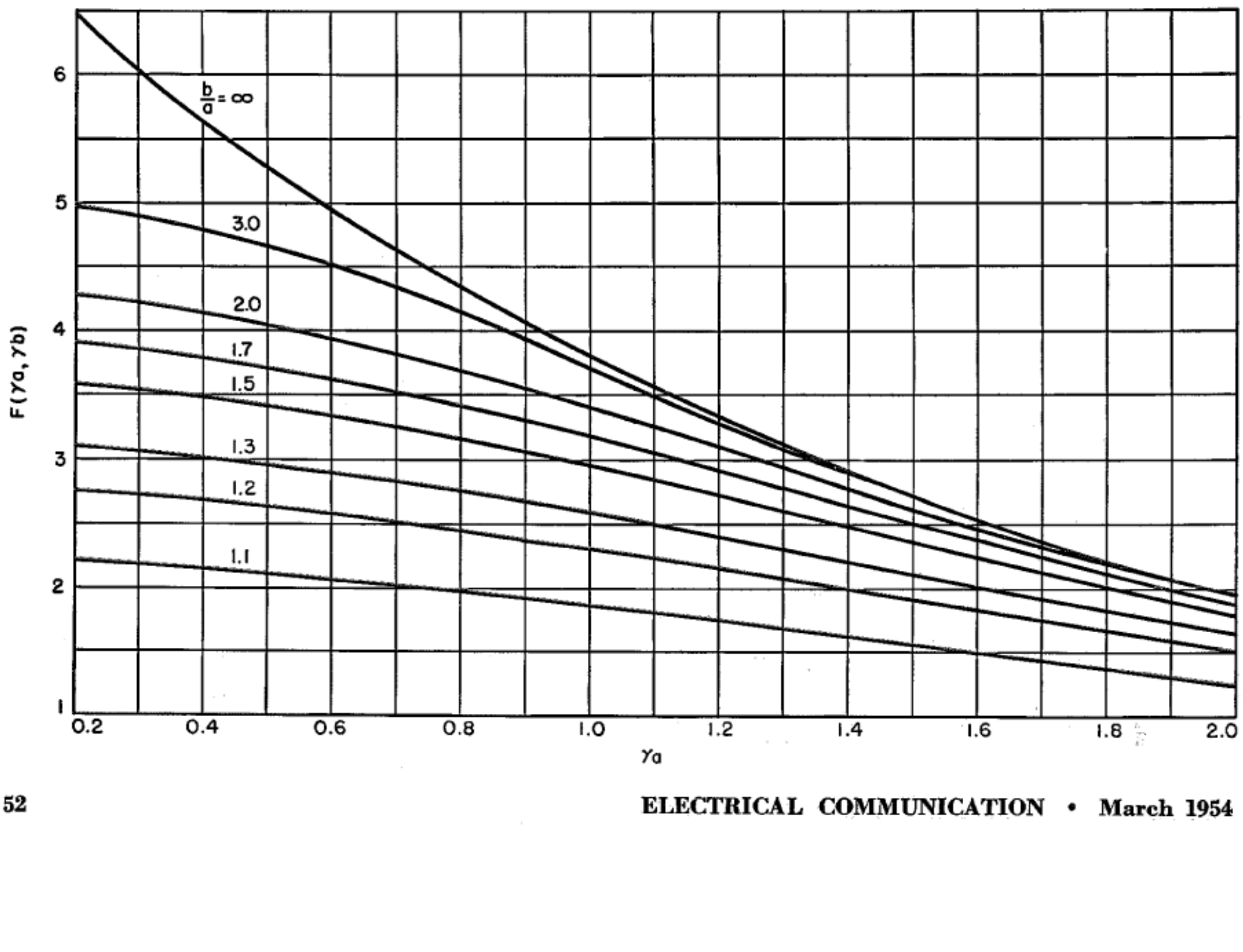


Figure 3—Above and below, impedance function $F(\gamma a, \gamma b)$ plotted against γa for a helically conductive cylinder within a coaxial conductive cylinder.



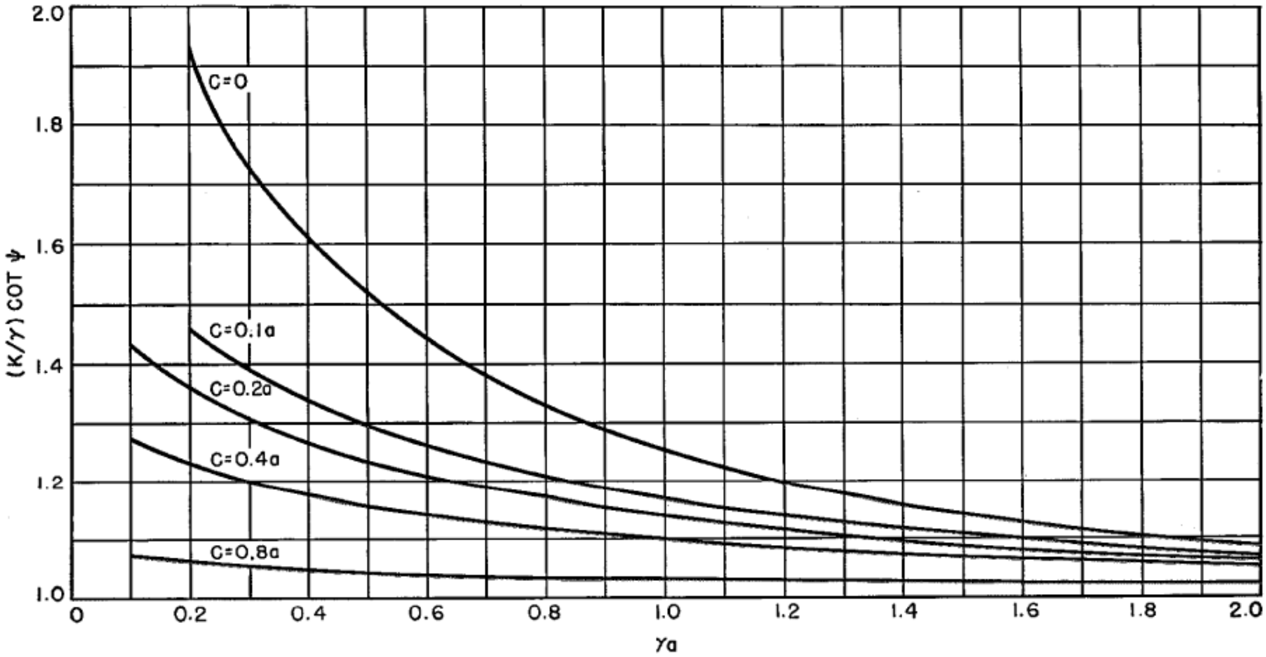


Figure 4—$(k/\gamma)\cot\psi$, a quantity proportional to velocity, plotted as a function of γa, a quantity proportional to frequency, for a helically conductive cylinder with an inner coaxial conductive cylinder.

constant is given[3] by

$$\left(\frac{k}{\gamma}\cot\psi\right)^2 = \frac{I_0(\gamma a)K_0(\gamma a)}{I_1(\gamma a)K_1(\gamma a)}\left[\frac{1-\dfrac{I_0(\gamma c)K_0(\gamma a)}{K_0(\gamma c)I_0(\gamma a)}}{1-\dfrac{I_1(\gamma c)K_1(\gamma a)}{K_1(\gamma c)I_1(\gamma a)}}\right]. \tag{5}$$

The solution is shown plotted in Figure 4 with b/a as a parameter.

4. *Helical Conductor Between Coaxial Conductive Cylinders*

For this condition, radius c is finite and less than a, while b is finite and greater than a. The radial propagation constant is given by

$$\left(\frac{k}{\gamma}\cot\psi\right)^2 = \frac{I_0(\gamma a)K_0(\gamma a)}{I_1(\gamma a)K_1(\gamma a)}\,\frac{I_0(\gamma b)K_1(\gamma b)}{I_1(\gamma b)K_0(\gamma b)}\,\frac{I_1(\gamma c)K_0(\gamma c)}{I_0(\gamma c)K_1(\gamma c)} \times\left[\frac{\left(1-\dfrac{I_0(\gamma a)K_0(\gamma b)}{K_0(\gamma a)I_0(\gamma b)}\right)\left(1-\dfrac{I_0(\gamma c)K_0(\gamma a)}{K_0(\gamma c)I_0(\gamma a)}\right)\left(1-\dfrac{I_1(\gamma b)K_1(\gamma c)}{I_1(\gamma c)K_1(\gamma b)}\right)}{\left(1-\dfrac{I_1(\gamma a)K_1(\gamma b)}{K_1(\gamma a)I_1(\gamma b)}\right)\left(1-\dfrac{I_1(\gamma c)K_1(\gamma a)}{K_1(\gamma c)I_1(\gamma a)}\right)\left(1-\dfrac{I_0(\gamma b)K_0(\gamma c)}{I_0(\gamma c)K_0(\gamma b)}\right)}\right]. \tag{6}$$

The solution is shown plotted in Figure 5 with $c:a:b$ as a parameter.

5. *Acknowledgments*

The form of the expression given in (4) is due to Mr. V. R. Saari. The computations for the curves were carried out by Mrs. E. J. White.

6. *Glossary of Symbols*

a = mean radius of helically conductive cylinder
b = inner radius of outer conductive cylinder
c = outer radius of inner conductive cylinder
c = velocity of light
E = electric field along the axis indicated by the subscript

[3] O. Doehler and W. Kleen, "Effect of the Transverse Electric Vector on the Delay Line of a Traveling-Wave Tube," *Annales de Radioélectricité*, volume 4, pages 117–130; 1949.

I = modified Bessel function of the first kind and of the order indicated by the subscript
$k = \omega/c$
K = modified Bessel function of the second kind and of the order indicated by the subscript
P = total power flowing
v = axial phase velocity
$\beta = \omega/v$
$\gamma = (\beta^2 - k^2)^{1/2}$ = radial propagation constant
ψ = pitch angle between helix and a circumference
$\omega = 2\pi f$ = radian frequency.

7. Appendix—Wave Propagation on a Helical Conductor Surrounded by a Coaxial Conductive Cylinder[4]

The problem of propagation on a helical conductor has been treated in detail.[1] In many applications of helical delay lines as well as traveling-wave tubes, the helix is surrounded by a conductive cylinder. We are interested in the effect of a uniformly conductive cylinder on the phase velocity of the wave and on the impedance parameter defined as

$$(E_z^2/\beta^2 P)^{1/3}. \tag{1}$$

The field components representing solutions of the wave equation in cylindrical coordinates for a plane wave having circular symmetry and propagating in the z direction with velocity

$$v = \omega/\beta \tag{2}$$

for the model of Figure 1, with the center conductor removed, with lossless conductors, and with space having a dielectric constant equal to that of vacuum are

Inside radius a, $r \leqslant a$,

$$H_{z1} = B_1 I_0(\gamma r) \tag{3}$$

$$E_{z3} = B_3 I_0(\gamma r) \tag{4}$$

$$H_{\phi 3} = B_3 \frac{j\omega\epsilon}{\gamma} I_1(\gamma r) \tag{5}$$

$$H_{r1} = B_1 \frac{j\beta}{\gamma} I_1(\gamma r) \tag{6}$$

$$E_{\phi 1} = - B_1 j \frac{\omega\mu}{\gamma} I_1(\gamma r) \tag{7}$$

$$E_{r3} = B_3 \frac{j\beta}{\gamma} I_1(\gamma r). \tag{8}$$

[4] Similar results for propagation constants using a different approach were obtained by W. Sichak in, "Coaxial Line with Helical Inner Conductor," soon to be published in *Proceedings of the IRE*.

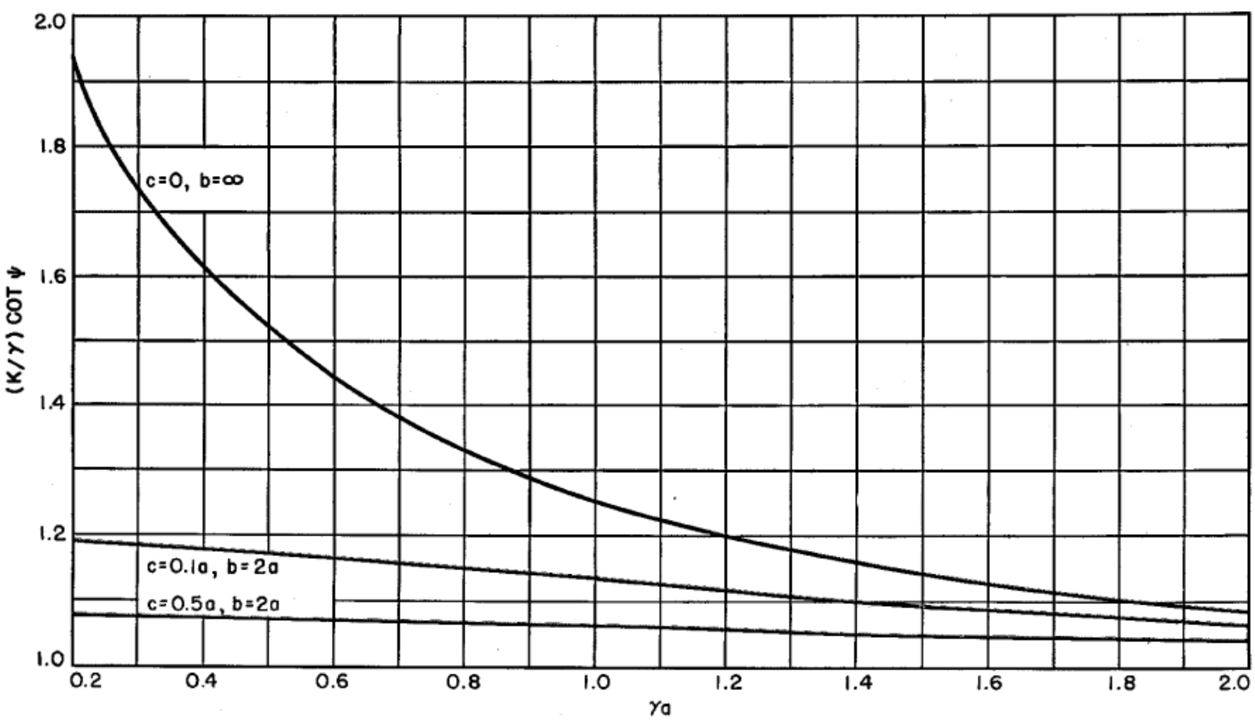


Figure 5—(k/γ) cot ψ, a quantity proportional to velocity, plotted as a function of γa, a quantity proportional to frequency, for a helically conductive cylinder between two coaxial conductive cylinders.

Outside radius a, $r \leqslant a \leqslant b$

$$H_{z2} = B_2[I_0(\gamma r)K_1(\gamma b) + K_0(\gamma r)I_1(\gamma b)] \quad (9)$$

$$E_{z4} = B_4[I_0(\gamma r)K_0(\gamma b) - K_0(\gamma r)I_0(\gamma b)] \quad (10)$$

$$H_{\phi 4} = -B_4 \frac{j\omega\epsilon}{\gamma} \times [I_1(\gamma r)K_0(\gamma b) + K_1(\gamma r)I_0(\gamma b)] \quad (11)$$

$$H_{r2} = -B_2 j \frac{\beta}{\gamma} \times [I_1(\gamma r)K_1(\gamma b) - K_1(\gamma r)I_1(\gamma b)] \quad (12)$$

$$E_{\phi 2} = B_2 j \frac{\omega\mu}{\gamma} \times [I_1(\gamma r)K_1(\gamma b) - K_1(\gamma r)I_1(\gamma b)] \quad (13)$$

$$E_{r4} = -B_4 j \frac{\beta}{\gamma} \times [I_1(\gamma r)K_0(\gamma b) + K_1(\gamma r)I_0(\gamma b)], \quad (14)$$

where

$$\left.\begin{aligned} \gamma &= (\beta^2 - k^2)^{1/2} \\ \beta &= \omega/v \\ k &= \omega/c \end{aligned}\right\} \quad (15)$$

and all field components [(3) through (14)] are multiplied by exp $[j(\omega t - \beta_z)]$.

The boundary conditions to be satisfied at the cylinder of radius a are as follows.

First,

$$E_{z3} \sin\psi + E_{\phi 1} \cos\psi = 0 \quad (16)$$

$$E_{z4} \sin\psi + E_{\phi 2} \cos\psi = 0. \quad (17)$$

Second,

$$E_{z3} = E_{z4} \quad (18)$$

$$E_{\phi 1} = E_{\phi 2}. \quad (19)$$

Third,

$$H_{z1} \sin\psi + H_{\phi 3} \cos\psi = H_{z2} \sin\psi + H_{\phi 4} \cos\psi. \quad (20)$$

These boundary conditions applied to (3) through (14) yield an expression for the determination of the propagation constant γ.

$$\left(\frac{k}{\gamma \cot\psi}\right)^2 = \frac{I_0(\gamma a)K_0(\gamma a)}{I_1(\gamma a)K_1(\gamma a)} \left[\frac{1 - \dfrac{I_0(\gamma a)K_0(\gamma b)}{K_0(\gamma a)I_0(\gamma b)}}{1 - \dfrac{I_1(\gamma a)K_1(\gamma b)}{K_1(\gamma a)I_1(\gamma b)}}\right]. \quad (21)$$

The field components in terms of a common amplitude factor B are as follows.

Inside the helix, $r \leqslant a$

$$E_z = BI_0(\gamma r) \quad (22)$$

$$E_r = jB \frac{\beta}{\gamma} I_1(\gamma r) \quad (23)$$

$$E_\phi = -B \frac{I_0(\gamma a)}{I_1(\gamma a)} \frac{1}{\cot\psi} I_1(\gamma r) \quad (24)$$

$$H_z = -j \frac{B}{K} \frac{\gamma}{k} \frac{I_0(\gamma a)}{I_1(\gamma a)} \frac{1}{\cot\psi} I_0(\gamma r) \quad (25)$$

$$H_r = \frac{B}{K} \frac{\beta}{k} \frac{I_0(\gamma a)}{I_1(\gamma a)} \frac{1}{\cot\psi} I_1(\gamma r) \quad (26)$$

$$H_\phi = j \frac{B}{K} \frac{k}{\gamma} I_1(\gamma r). \quad (27)$$

Outside the helix, $a \leqslant r \leqslant b$

$$E_z = B\, I_0(\gamma a) \times \left[\frac{I_0(\gamma r)K_0(\gamma b) - K_0(\gamma r)I_0(\gamma b)}{I_0(\gamma a)K_0(\gamma b) - K_0(\gamma a)I_0(\gamma b)}\right] \quad (28)$$

$$E_r = jB \frac{\beta}{\gamma} I_0(\gamma a) \times \left[\frac{I_1(\gamma r)K_0(\gamma b) + K_1(\gamma r)I_0(\gamma b)}{I_0(\gamma a)K_0(\gamma b) - K_0(\gamma a)I_0(\gamma b)}\right] \quad (29)$$

$$E_\phi = -BI_0(\gamma a) \frac{1}{\cot\psi} \times \left[\frac{I_1(\gamma r)K_1(\gamma b) - K_1(\gamma r)I_1(\gamma b)}{I_1(\gamma a)K_1(\gamma b) - K_1(\gamma a)I_1(\gamma b)}\right] \quad (30)$$

$$H_z = -j \frac{B}{K} \frac{\gamma}{k} I_0(\gamma a) \frac{1}{\cot\psi} \times \left[\frac{I_0(\gamma r)K_1(\gamma b) + K_0(\gamma r)I_1(\gamma b)}{I_1(\gamma a)K_1(\gamma b) - K_1(\gamma a)I_1(\gamma b)}\right] \quad (31)$$

$$H_r = \frac{B}{K} \frac{\beta}{k} I_0(\gamma a) \frac{1}{\cot\psi} \times \left[\frac{I_1(\gamma r)K_1(\gamma b) - K_1(\gamma r)I_1(\gamma b)}{I_1(\gamma a)K_1(\gamma b) - K_1(\gamma a)I_1(\gamma b)}\right] \quad (32)$$

$$H_\phi = -j \frac{B}{K} \frac{k}{\gamma} I_0(\gamma a) \times \left[\frac{I_1(\gamma r)K_0(\gamma b) + K_1(\gamma r)I_0(\gamma b)}{I_0(\gamma a)K_0(\gamma b) - K_0(\gamma a)I_0(\gamma b)}\right], \quad (33)$$

where

$$K = (\mu/\epsilon)^{1/2} = 120\pi, \text{ ohms} \quad (34)$$

and all field components [(22) through (33)] are multiplied by exp $[j(\omega t - \beta_z)]$.

The impedance parameter may now be evaluated by use of these field components. The power associated with the propagation is given by

$$P = \tfrac{1}{2}R^{e} \int E \times H^{*}\, d\tau \tag{35}$$

taken over a plane normal to the axis of propagation. This is

$$P = \pi R_{e}\left[\int_{0}^{a} (E_{r}H_{\phi}{}^{*} - E_{\phi}H_{r}{}^{*}) r dr + \int_{a}^{b} (E_{r}H_{\phi}{}^{*} - E_{\phi}H_{r}{}^{*}) r dr\right]. \tag{36}$$

This yields

$$\begin{aligned} P = E_{z}{}^{2}(0)\,\frac{\pi}{2K}\frac{\beta a k a}{\gamma^{2}}\Bigg\{\left[1 + \frac{I_{0}{}^{2}(\gamma a)}{I_{1}{}^{2}(\gamma a)R}\right]\left[\frac{a^{2}}{2}M(\gamma a)\right] + \left[HK_{0}{}^{2}(\gamma b) + \frac{JK_{1}{}^{2}(\gamma b)}{R}\right] \\ \times\left[\frac{b^{2}}{2}M(\gamma b) - \frac{a^{2}}{2}M(\gamma a)\right] + \left[HI_{0}{}^{2}(\gamma b) + \frac{JK_{1}{}^{2}(\gamma b)}{R}\right]\left[\frac{b^{2}}{2}N(\gamma b) - \frac{a^{2}}{2}N(\gamma a)\right] \\ + 2\left[HI_{0}(\gamma b)K_{0}(\gamma b) - \frac{J}{R}I_{1}(\gamma b)K_{1}(\gamma b)\right]\left[\frac{b^{2}}{2}P(\gamma b) - \frac{a^{2}}{2}P(\gamma b)\right]\Bigg\}. \end{aligned} \tag{37}$$

Defining the impedance function as

$$(E_{z}{}^{2}/\beta^{2}P)^{1/3} = (\beta/k)^{1/3}(\gamma/\beta)^{4/3}F(\gamma a,\gamma b) \tag{38}$$

gives

$$\begin{aligned} F(\gamma a,\gamma b) = \Bigg\{\frac{(\gamma a)^{2}}{240}\Bigg\{\left[1 + \frac{I_{0}{}^{2}(\gamma a)}{RI_{1}{}^{2}(\gamma a)}\right]\left[M(\gamma a)\right] + I_{0}{}^{2}(\gamma a)\Bigg[\left(\frac{K_{0}{}^{2}}{H} + \frac{K_{1}{}^{2}}{JR}\right)\left(\frac{b^{2}}{a^{2}}M(\gamma b) - M(\gamma a)\right) \\ + \left(\frac{I_{0}{}^{2}}{H} + \frac{I_{1}{}^{2}}{JR}\right)\left(\frac{b^{2}}{a^{2}}N(\gamma b) - N(\gamma a)\right) + 2\left(\frac{I_{0}K_{0}}{H} - \frac{I_{1}K_{1}}{JR}\right)\left(\frac{b^{2}}{a^{2}}P(\gamma b) - P(\gamma a)\right)\Bigg]\Bigg\}\Bigg\}^{-1/3}. \end{aligned} \tag{39}$$