A SIMPLIFIED PRECISION FORMULA FOR THE INDUCTANCE OF A HELIX WITH CORRECTIONS FOR THE LEAD-IN WIRES

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ABSTRACT

A precision formula is here given for the self inductance of a single-layer helix wound with ordinary round wire. This is neither more nor less accurate than that previously published, from which it has been derived by evaluating certain correction terms and replacing them by approximation formulas which are much simpler to compute.

CONTENTS

I. INTRODUCTION

The formula derived in this paper is a modification, without loss of precision, of the expression for the inductance of a single-layer helix of round wire, given in Bureau of Standards Scientific Paper No. 537 (vol. 21, p. 431, 1926–27). The notation has been changed, and the correction terms transformed so that the present formula is more simple from the point of view of the computer.

The diameter of the wire is d; the mean diameter of the solenoid is D; and its length l is the axial distance from the center of the wire at the begininng of the first turn to the center of the wire at the end of the N^{th} turn, N being the total turns so that the pitch of the winding is l/N. The modulus k of the complete ellipitic integrals K and E is given by

$$k^2=\frac{D^2}{l^2+D^2}$$

The principal term L_s is the current-sheet formula of Lorenz.

$$L_s=\frac{4\pi N^2}{3}\sqrt{l^2+D^2}\left[K-E+\frac{D^2}{l^2}\ (E-k)\right] \tag{1}$$

As one of the correction terms there appears M, the mutual inductance between the two end-circles of the solenoid, which is computed by

$$M=4\pi D\left[\frac{K-E}{k}-\frac{k}{2}K\right] \tag{2}$$

The inductance L of an actual helix is (to a precision which neglects terms of the order of $\left(\frac{l}{ND}\right)^3\log\frac{l}{ND}$)

$$L = L_s + \pi D\left[2N\left(ln\frac{l}{Nd} - 0.89473\right) + \frac{1}{3}ln\frac{N\pi D}{l}\right] + lP\left(\frac{l}{D}\right)$$
$$-\frac{1}{6}M - \frac{2}{\pi}\sqrt{l^2+D^2}\,(E-k)\left[2 \pm \left(\frac{N\pi d}{2l}\right)^2\right] \qquad (3)$$
$$-\frac{l\pi}{2}\left[1 - \frac{l}{D}sin^{-1}\frac{D}{\sqrt{l^2+D^2}}\right]$$

where

$$P(\eta) = \frac{1}{4\eta^2} + 2ln4\eta \text{ when } \eta \geqq 1 \qquad (4)$$
$$= 3\eta - \eta ln\eta \text{ when } \eta \leqq 1$$

The ambiguous sign is plus for the "natural" and minus for the uniform distribution of current over the section of the wires. A current density inversely proportional to the distance from the axis of the solenoid is called the natural distribution.

The formula for L here given is obtained by change of notation and by transformation of the function A_2 (k) in formula (114) of the paper cited. (This formula contains a misprint in the term $\frac{1}{3}\log\frac{\alpha}{p}$ which should be $\frac{1}{3}\log\frac{a}{p}$). In the present notation this formula is

$$L = L_s + \pi D\left[2N\left(ln\frac{l}{Nd} - 0.89473\right) + \frac{1}{3}ln\frac{N\pi D}{l}\right]$$
$$+\pi DA_2(k) \mp \frac{2}{\pi}\sqrt{l^2+D^2}(E-k)\left(\frac{N\pi d}{2l}\right)^2 \qquad (A)$$

The function $A_2(k)$ is defined by formula (89)

$$A_2(k) = \frac{4}{\pi^2}[B_0(k) - B_1(k) + B_2] + 0.66267 - \frac{1}{3}ln\pi$$

The numerical quantity B_2 was defined in (90) as a series which has been found to be -0.60835. Hence

$$\pi DA_2(k) = D\left[\frac{4}{\pi}(B_0(k) - B_1(k)) + 0.11\right] \qquad (5)$$

The term 0.11 in the parenthesis will be omitted since 0.1 D will be considered negligible in this small correction term. As most solenoids to be used as precision standards would have an inductance greater than 10 millihenries (10^7 cm), it is evident that with a diameter of say 30 cm, 0.1 D would be 3 cm, which is about 3 parts in 10,000,000. Hence, in the transformations of $B_0(k)$ and $B_1(k)$ which follow, terms of the order of 0.1 may be considered negligible.

II. TRANSFORMATION OF $B_0(k)$

On page 507 of the paper quoted, B_0 is given by

$$B_0(k) = 1 - \frac{E(k)}{k} + \frac{\sqrt{1-k^2}}{k}\int_k^1 \frac{K(x)dx}{x\sqrt{1-x^2}} \qquad (6)$$

If we let

$$\eta=\frac{\sqrt{1-k^2}}{k} \text{ and } x^2=\frac{1}{1+\lambda^2} \text{ and}$$

$$P(\eta)\equiv\frac{4}{\pi}\int_k^1\frac{K(x)dx}{x\sqrt{1-x^2}}=\frac{4}{\pi}\int_0^\eta\frac{d\lambda}{\sqrt{1+\lambda^2}}K\left(\frac{1}{\sqrt{1+\lambda^2}}\right)$$

$$=\frac{4}{\pi}\int_0^{\frac{\pi}{2}}d\theta\int_0^{\frac{\pi}{2}}\frac{d\lambda}{\sqrt{\lambda^2+sm^2\theta}}=\frac{4}{\pi}\int_0^{\frac{\pi}{2}}\log\left[\frac{\eta+\sqrt{\eta^2+\sin^2\theta}}{\sin\theta}\right]d\theta \tag{7}$$

then (6) gives

$$\frac{4D}{\pi}B_0(k)=-\frac{4}{\pi}\sqrt{l^2+D^2}[E(k)-k]+lP\left(\frac{l}{d}\right) \tag{8}$$

We may evaluate $P(\eta)$ first for the case where $\eta\geqq1$; that is, where $l\geqq D$. Using the formula

$$\int_0^{\frac{\pi}{2}}\log\sin\theta\,d\theta=-\frac{\pi}{2}\log 2$$

we may write (7) in the form

$$P(\eta)=2\log 2+\frac{4}{\pi}\int_0^{\frac{\pi}{2}}\log(\eta+\sqrt{\eta^2+\sin^2\theta})\,d\theta$$

$$=2\log 2\eta+\frac{4}{\pi}\int_0^{\frac{\pi}{2}}\log\left(1+\sqrt{1+\frac{\sin^2\theta}{\eta^2}}\right)d\theta$$

or since (for $\eta\geqq1$)

$$\log\left(1+\sqrt{1+\frac{\sin^2\theta}{\eta^2}}\right)=\log 2-\frac{1}{2\sqrt{\pi}}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(n+1)}\frac{\sin^{2n}\theta}{\eta^{2n}}$$

$$P(\eta)=2\log 4\eta+\frac{1}{4\eta^2}-\sum_{n=2}^{\infty}\frac{(-1)^n}{n}\left[\frac{1.3.5\ .\ .\ (2n-1)}{2.4.6\ .\ .\ .\ 2n}\right]^2\frac{1}{\eta^{2n}} \text{ if } \eta\geqq1 \tag{9}$$

Since the sum of this alternating series is numerically less than the value of its first term $\frac{9}{128\eta^4}$, which is never greater than about $\frac{1}{14}$ if $\eta\geqq1$, it is sufficient to take

$$P(\eta)=2\log 4\eta+\frac{1}{4\eta^2} \qquad \text{when } \eta\geqq1 \tag{10}$$

When $\eta=1$ this becomes $2\log 4+\frac{1}{4}=3.02$ while the exact expression (9) becomes 2.98. We may, therefore, take $P(1)=3$. On the other hand, when $0\leqq\eta\leqq1$, the approximation

$$P(\eta)=3\eta+\eta\log\frac{1}{\eta} \qquad \text{when } 0\leqq\eta\leqq1 \tag{10$'$}$$

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will be in error by not more than 0.1. This may be seen from the fact that (10)′ is correct at $\eta=0$ and $\eta=1$, while the error between 0 and 1 is

$$Y(\eta)\equiv 3\eta+\eta\log\frac{1}{\eta}-P(\eta)$$

which has its greatest numerical value when

$$\eta=\frac{\sqrt{1-k^2}}{k}=0.34904$$

corresponding to $k^2=0.8914$. This is the root of the equation

$$Y'(\eta)=0=2+\log\frac{1}{\eta}-P'(\eta)=2+\frac{1}{2}\log\frac{k^2}{1-k^2}-\frac{4k}{\pi}K_{(k)}$$

To estimate this greatest error, we may obtain an upper and lower limit to $P(\eta)$ as follows:
Write

$$P(\eta)=\log 4+\frac{1}{\pi}\int_0^\pi \log\,(\eta+\sqrt{1+\eta^2-\cos^2\theta})^2 d\theta$$

$$=\log 4+\frac{1}{\pi}\int_0^\pi \log\,(1+2\eta^2-\cos^2\theta+2\eta\sqrt{1+\eta^2-\cos^2\theta})\,d\theta$$

Now

$$2\eta\sqrt{1+\eta^2-\cos^2\theta}=2\eta\sqrt{1+\eta^2}-\frac{\eta}{\sqrt{1+\eta^2}}\cos^2\theta-Z\,(\eta,\,\theta)$$

where

$$Z\,(\eta,\,\theta)=\frac{\eta\sqrt{1+\eta^2}}{\sqrt{\pi}}\sum_{s=2}^{\infty}\frac{\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s+1)}\,k^{2s}\cos^{2s}\theta$$

where

$$k^2=\frac{1}{1+\eta^2}$$

The function Z is never negative. It vanishes when $\theta=\frac{\pi}{2}$ and has its greatest value when $\theta=o$ and $\theta=\pi$ which is

$$Z\,(\eta,\,\theta)=\frac{\eta(\sqrt{1+\eta^2}-\eta)^2}{\sqrt{1+\eta^2}}$$

Hence $P(\eta)$ may be written in the form

$$P(\eta)=\frac{1}{\pi}\int_s^\pi \log 4\left\{(\sqrt{1+\eta^2}+\eta)^2-Z(\eta,\,\theta)-\left(\frac{\sqrt{1+\eta^2}+\eta}{\sqrt{1+\eta^2}}\right)\cos^2\theta\right\}d\theta$$

If in this integral we replace the variable function $Z(\eta,\,\theta)$ by its smallest value, zero, we get a function $F(\theta)$, which is an upper limit for $P(\eta)$ for all values of η.

$$F(\eta)\equiv\frac{1}{\pi}\int_s^\pi \log 4\left\{(\sqrt{1+\eta^2}+\eta)^2-\left(\frac{\sqrt{1+\eta^2}+\eta}{\sqrt{1+\eta^2}}\right)\cos^2\theta\right\}d\theta$$

If we replace $Z(\eta,\theta)$ by its greatest value, we get a function $f(\eta)$ which is a lower limit to $P(\eta)$ for all values of (η) where

$$f(\eta)\equiv\frac{1}{\pi}\int_{\delta}^{\pi}\log 4\left\{(\sqrt{1+\eta^2}+\eta)^2-\frac{\eta(\sqrt{1+\eta^2}-\eta)^2}{\sqrt{1+\eta^2}}-\left(\frac{\sqrt{1+\eta^2}+\eta}{\sqrt{1+\eta^2}}\right)\cos^2\theta\right\}d\theta$$

That is $f(\theta) < P(\eta) < F(\eta)$ if $\eta > 0$.

Both of these functions may be evaluated by means of the definite integral formula

$$\frac{1}{\pi}\int_0^{\pi}\log 4\ (a^2-b^2\cos^2\theta)\ d\theta=2\log\ (a+\sqrt{a^2-b^2})\text{ if } a\geqq b$$

which gives

$$F(\eta)=2\log\ (\sqrt{1+\eta^2}+\eta)\left(1+\sqrt{\frac{\eta}{\sqrt{1+\eta^2}}}\right)$$

$$f\ (\eta)=2\log\left(2\eta+\sqrt{1+4\eta^2+\frac{\eta}{\sqrt{1+\eta^2}}}\right)$$

This gives

$$F(0)=0,\ F(1)=2.984$$

$$f(0)=0,\ f(1)=2.958$$

At the point of maximum error in the range $0<\eta<1$, which is at $\eta_1=0.349$, we find

$$\left.\begin{array}{l}F(\eta_1)=1.581\\ f(\eta_1)=1.431\end{array}\right\}\text{whence } P(\eta_1)=1.506\ \pm 0.075$$

and

$$3\ \eta_1+\eta_1\log\frac{1}{\eta_1}=1.415$$

which shows that the maximum error made by using (10)′ for $P(\eta)$ is approximately 0.1 in the range $0\leqq\eta\leqq 1$.

III. TRANSFORMATION OF $B_1(k)$

The function $B_1(k)$ is defined in the paper quoted, on page 453, by

$$kB_1(k)=\frac{\pi^2}{12}\ E(k)-\int_0^{\frac{\pi}{2}}\theta^2\sqrt{1-k^2\sin^2\theta}\,d\theta$$
$$-\left(1-\frac{k^2}{2}\right)\left[\frac{\pi^2}{12}K(k)-\int_0^{\frac{\pi}{2}}\frac{\theta^2 d\theta}{\sqrt{1-k^2\sin^2\theta}}\right]$$

Now

$$\int_0^{\frac{\pi}{2}}\theta^2\sqrt{1-k^2\sin^2\theta}\,d\theta=\frac{\pi^2}{4}E(k)-\int_0^{\frac{\pi}{2}}\phi(\pi-\phi)\sqrt{1-k^2\cos^2\phi}\,d\phi$$

and

$$\int_0^{\frac{\pi}{2}}\frac{\theta^2 d\theta}{\sqrt{1-k^2\sin^2\theta}}=\frac{\pi^2}{4}K(k)-\int_0^{\frac{\pi}{2}}\frac{\phi(\pi-\phi)}{\sqrt{1-k^2\cos^2\phi}}d\phi$$

Hence

$$\frac{4D}{\pi}B_1(k)=\frac{4\pi D}{6}\left[\frac{K(k)-E(k)}{k}-\frac{kK(k)}{2}\right]+DU(k)$$

where

$$U(k)\equiv -\frac{\pi k}{2}\int_0^{\frac{\pi}{2}} \frac{4\phi(\pi-\phi)}{\pi^2}\,\frac{\cos 2\phi}{\sqrt{1-k^2\cos^2\phi}}\,d\phi$$

The factor of the integrand $y(\phi)=\frac{4}{\pi^2}\phi\ (\pi-\phi)$ is represented by a parabola which vanishes when $\phi=0$ and has the maximum value unity when $\phi=\frac{\pi}{2}$, where it also has a horizontal slope. It is therefore very approximately the same as $\sin\phi$. Replacing it by $\sin\phi$ gives the sufficient approximation

$$U(k)=-\frac{\pi k}{2}\int_0^{\frac{\pi}{2}}\frac{\cos 2\phi\sin\phi}{\sqrt{1-k^2\cos^2\phi}}\,d\phi=\frac{\pi\sqrt{1-k^2}}{2k}\left[1-\frac{\sqrt{1-k^2}}{k}\sin^{-1}k\right]$$

$$=\frac{\pi l}{2D}\left[1-\frac{l}{D}\sin^{-1}\frac{D}{\sqrt{l^2+D^2}}\right]$$

Hence

$$\frac{4D}{\pi}B_1(k)=\frac{M}{6}+\frac{l\pi}{2}\left[1-\frac{l}{D}\sin^{-1}\frac{D}{\sqrt{l^2+D^2}}\right] \qquad (11)$$

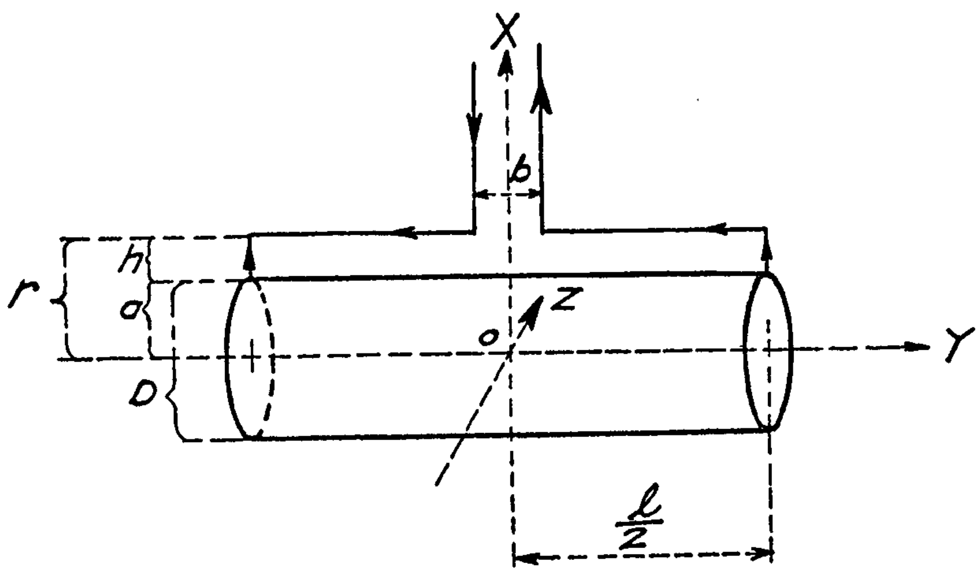

FIGURE 1.—*Arrangements of lead-in wires*

where M is defined by equation (2) and is obviously the mutual inductance between the two end circles of the solenoid. By use of equations (10) or (10)′ and (11), the equation (A) is transformed into formula (3) of this paper.

IV. CORRECTIONS DUE TO LEAD-IN WIRES

The inductance of the helix and lead-in wires together is

$$L'=L+L_1+2M_{hl}$$

where L is that of the helix given by formula (3), L_1 is the self-inductance of the lead-in wires and M_{hl} the mutual inductance between these wires and the helix. In computing M_{hl}, the helix may be considered a current sheet, and only its axial component of current contributes to M_{hl}. If the leads are disposed as shown in Figure 1, it is evident that these leads which are perpendicular to the axis of the cylinder have no mutual inductance with the cylinder. The latter carries unit current in the y = direction so that the current density is $\frac{1}{2\pi a}=\frac{1}{\pi D}$. Consequently, the term M_{hl} is given by

$$M_{hl}=2\int_0^{b/2}A(r,\ y)dy-2\int_0^{l/2}A(r,\ y)dy \tag{12}$$

where

$$A\ (r,\ y)=\frac{1}{2\pi}\int_{-l/2}^{l/2}dy'\int_0^{2\pi}\frac{d\ \phi}{\sqrt{(y-y')^2+a^2+r^2-2\ a\ r\cos\phi}}$$

$$=\frac{2}{\pi}\int_{-l/2}^{l/2}dy'\int_0^{\pi/2}\frac{d\ \theta}{\sqrt{(y-y')^2+(a+r)^2-4\ a\ r\sin^2\theta}}$$

$$=\frac{2}{\pi}\int_0^{\frac{l}{2}+y}dz\int_0^{\pi/2}\frac{d\ \theta}{\sqrt{z^2+(a+r)^2-4\ a\ r\sin^2\theta}}$$

$$+\frac{2}{\pi}\int_0^{\frac{l}{2}-y}dz\int_0^{\pi/2}\frac{d\ \theta}{\sqrt{z^2+(a+r)^2-4\ a\ r\sin^2\theta}}$$

or

$$A\ (r,\ y)=\frac{2}{\pi}\int_{k_y}^1\frac{d\ x}{x\sqrt{1-x^2}}K\left(\frac{2\sqrt{ar}}{a+r}x\right)+\frac{2}{\pi}\int_{k_{-y}}^1\frac{d\ x}{x\sqrt{1-x^2}}K\left(\frac{2\sqrt{ar}}{a+r}x\right) \tag{13}$$

where

$$\frac{1}{k_y^2}=1+\left(\frac{\frac{l}{2}+y}{a+r}\right)^2 \tag{14}$$

As a special case of this

$$A\ (a,\ o)=\frac{4}{\pi}\int_{k_o}^1\frac{K\ (x)\ d\ x}{x\sqrt{1-x^2}}=P\ (\eta_0)\text{ by (7)} \tag{15}$$

Where

$$k_0^2=\frac{1}{1+\left(\frac{l}{4\ a}\right)^2}=\frac{1}{1+\left(\frac{l}{2\ D}\right)^2}\text{ and }\eta_0=\frac{l}{2\ D} \tag{16}$$

Now since b and $r-a=h$ are both small, we may write

$$2\int_0^{b/2}A(r,y)dy=2A(a,0)\int_0^{b/2}dy=bA(a,0)=bP(\eta_0) \tag{17}$$

In the first integral of (12) we may write

$$A(ry)=A(a+h,y)=A(a,y)+h\left[\frac{\partial A(r,y)}{\partial r}\right]_{r=a}$$

Computing $\left[\frac{\partial A(ry)}{\partial r}\right]_{r=a}$ from (13), since $\left[\frac{\partial}{\partial r}\frac{2\sqrt{ar}}{a+r}\right]_{r=a}=0,$

one obtains after placing $r=a$

$$\frac{\partial k_y}{\partial r}=\frac{k_y(1-k^2_y)}{D} \text{ and } \frac{\partial k_{-y}}{\partial r}=\frac{k_{-y}(1-k^2_{-y})}{D}$$

so that

$$-2h\int_0^{l/2}\left[\frac{\partial A(r,y)}{\partial r}\right]_{r=a}dy=\frac{4h}{\pi D}\left[\int_0^{l/2}\sqrt{1-k^2_y}\,K(k_y)dy+\int_0^{l/2}\sqrt{1-k^2_{-y}}\,K(k_{-y})dy\right]$$

where

$$k^2_y=\frac{1}{1+\left(\frac{l/2+y}{D}\right)^2}$$

Changing the variable of integration from y to k_y gives

$$-2h\int_0^{l/2}\left[\frac{\partial A(ry)}{\partial r}\right]_{r=a}dy=\frac{4h}{\eta}\left[-\int_{k_0}^{k}\frac{K(x)dx}{x^2}+\int_{k_0}^{1}\frac{K(x)dx}{x^2}\right]=\frac{4h}{\pi}\int_k^1\frac{K(x)dx}{x^2}$$

$$=\frac{4h}{\pi k}\left[E(k)-k\right] \text{ where } k^2=\frac{D^2}{l^2+D^2} \qquad (18)$$

Similarly

$$-2\int_0^{l/2}A\,(a,\,y)\,dy=-\frac{4}{\pi}\left[\int_0^{l/2}dy\int_{k_y}^1\frac{K(x)dx}{x\sqrt{1-x^2}}+\int_0^{l/2}dy\int_{k_y}^1\frac{K(x)dx}{x\sqrt{1-x^2}}\right]$$

$$=-\frac{4D}{\pi}\int_k^1\frac{dz}{z^2\sqrt{1-z^2}}\int_z^1\frac{K(x)dx}{x\sqrt{1-x^2}}=-\frac{4D}{\pi}B_0(k) \qquad (19)$$

Adding (17), (18), and (19) gives by (12)

$$M_{hl}=\frac{4}{\pi}\left[-DB_0(k)+\frac{h(E(k)-k)}{k}\right]+bP(\eta_0) \qquad (20)$$

where

$$k^2=\frac{1}{1+\frac{l^2}{D^2}} \text{ and } k_0{}^2=\frac{1}{1+\left(\frac{l}{2D}\right)^2}$$

By use of (8) this may be put in the general form

$$M_{hl}=\left(1+\frac{h}{D}\right)\sqrt{l^2+D^2}\cdot\frac{4}{\pi}\left[E-k\right]-lP\left(\frac{l}{D}\right)+bP\left(\frac{l}{2D}\right) \qquad (21)$$

where $P(x)$ is defined by (4).

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