[From the *Transactions of the Royal Society of Edinburgh*, Vol. XXVI.]

XLVIII. *On the Geometrical Mean Distance of Two Figures on a Plane.*

[Received January 5th; read January 15th, 1872.]

THERE are several problems of great practical importance in electro-magnetic measurements, in which the value of a quantity has to be calculated by taking the sum of the logarithms of the distances of a system of parallel wires from a given point. The calculation is in some respects analogous to that in which we find the potential at a point due to a given system of equal particles, by adding the reciprocals of the distances of the particles from the given point. There is this difference, however, that whereas the reciprocal of a line is completely defined when we know the unit of length, the logarithm of a line has no meaning till we know not only the unit of length, but the modulus of the system of logarithms.

In both cases, however, an additional clearness may be given to the statement of the result by dividing, by the number of wires in the first case, and by the number of particles in the second. The result in the first case is the logarithm of a distance, and in the second it is the reciprocal of a distance; and in both cases this distance is such that, if the whole system were concentrated at this distance from the given point, it would produce the same potential as it actually does.

In the first case, since the logarithm of the resultant distance is the arithmetical mean of the logarithms of the distances of the various components of the system, we may call the resultant distance the geometrical mean distance of the system from the given point.

In the second case, since the reciprocal of the resultant distance is the arithmetical mean of the reciprocals of the distances of the particles, we may call the resultant distance the harmonic mean distance of the system from the given point.

The practical use of these mean distances may be compared with that of several artificial lines and distances which are known in Dynamics as the radius of gyration, the length of the equivalent simple pendulum, and so on. The result of a process of integration is recorded, and presented to us in a form which we cannot misunderstand, and which we may substitute in those elementary formulæ which apply to the case of single particles. If we have any doubts about the value of the numerical co-efficients, we may test the expression for the mean distance by taking the point at a great distance from the system, in which case the mean distance must approximate to the distance of the centre of gravity.

Thus it is well known that the harmonic mean distance of two spheres, each of which is external to the other, is the distance between their centres, and that the harmonic mean distance of any figure from a thin shell which completely encloses it is equal to the radius of the shell.

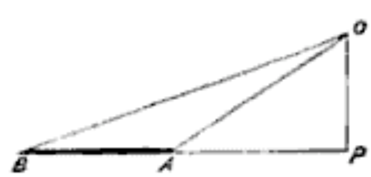
I shall not discuss the harmonic mean distance, because the calculations which lead to it are well known, and because we can do very well without it. I shall, however, give a few examples of the geometric mean distance, in order to shew its use in electro-magnetic calculations, some of which seem to me to be rendered both easier to follow and more secure against error by a free use of this imaginary line.

If the co-ordinates of a point in the first of two plane figures be x and y, and those of a point in the second ξ and η, and r denote the distance between these points, then R, the geometrical mean distance of the two figures, is defined by the equation

$$\log R \,.\, \iiiint dx\,dy\,d\xi\,d\eta = \iiiint \log r\, dx\,dy\,d\xi\,d\eta.$$

The following are some examples of the results of this calculation:—

(1) Let AB be a uniform line, and O a point not in the line, and let OP be the perpendicular from O on the line AB, produced if necessary, then if R is the geometric mean distance of O from the line AB,



$$AB\,.\,(\log R + 1) = PB\,.\log OB - PA \log OA + OP\,.\,A\hat{O}B.$$

(2) The geometrical mean distance of P, a point in the line itself, from AB is found from the equation

$$AB\,(\log R+1)=PB\log PB-PA\log PA.$$

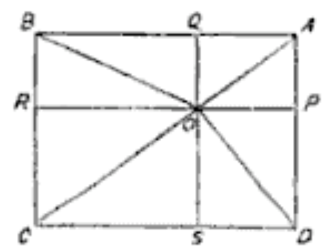
When P lies between A and B, PA must be taken negative, but in taking the logarithm of PA we regard PA as a positive numerical quantity.

(3) If R is the geometric mean distance between two finite lines AB and CD, lying in the same straight line,

$$AB\,.\,CD\,(2\log R+3)=AD^2\log AD+BC^2\log BC-AC^2\log AC-BD^2\log BD.$$

(4) If AB coincides with CD, we find for the geometric mean distance of all the points of AB from each other

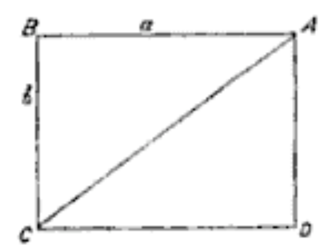
$$R=ABe^{-\frac{3}{2}}.$$



(5) If R is the geometric mean distance of the rectangle $ABCD$ from the point O in its plane, and POR and QOS are parallel to the sides of the rectangle through O,

$$\begin{aligned}ABCD\,(2\log R+3)=\;&2OP\,.\,OQ\log OA+2OQ\,.\,OR\log OB\\&+2OR\,.\,OS\log OC+2OS\,.\,OP\log OD\\&+OP^2\,.\,D\hat{O}A\qquad+OQ^2\,.\,A\hat{O}B\\&+OR^2\,.\,B\hat{O}C\qquad+OS^2\,.\,COD\end{aligned}$$

(6) If R is the geometric mean of the distances of all the points of the rectangle $ABCD$ from each other,



$$\log R=\log AC-\tfrac{1}{6}\frac{AB^2}{BC^2}\log\frac{AC}{AB}-\tfrac{1}{6}\frac{BC^2}{AB^2}\log\frac{AC}{BC}$$

$$+\tfrac{2}{3}\frac{AB}{BC}B\hat{A}C+\tfrac{2}{3}\frac{BC}{AB}A\hat{C}B-\tfrac{25}{12}.$$

When the rectangle is a square, whose side $=a$,

$$\begin{aligned}\log R&=\log a+\tfrac{1}{3}\log 2+\frac{\pi}{3}-\tfrac{25}{12}\\&=\log a-0\cdot8050866,\\R&=0\cdot44705\,a.\end{aligned}$$

(7) The geometric mean distance of a circular line of radius a, from a point in its plane at a distance r from the centre, is r if the point be without the circle, and a if the point be within the circle.

(8) The geometric mean distance of any figure from a circle which completely encloses it is equal to the radius of the circle. The geometric mean distance of any figure from the annular space between two concentric circles, both of which completely enclose it, is R, where

$$(a_1^2-a_2^2)(\log R+\tfrac{1}{2})=a_1^2\log a_1-a_2^2\log a_2,$$

a_1 being the radius of the outer circle, and a_2 that of the inner. The geometric mean distance of any figure from a circle or an annular space between two concentric circles, the figure being completely external to the outer circle, is the geometric mean distance of the figure from the centre of the circle.

(9) The geometric mean distance of all the points of the annular space between two concentric circles from each other is R, where

$$(a_1^2-a_2^2)^2(\log R-\log a_1)=\tfrac{1}{4}(3a_2^2-a_1^2)(a_1^2-a_2^2)-a_2^4\log\frac{a_1}{a_2}.$$

When a_2, the radius of the inner circle, vanishes, we find

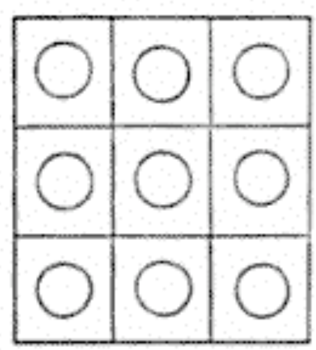
$$R=ae^{-\frac{1}{4}}.$$

When a_2, the radius of the inner circle, becomes nearly equal to a_1, that of the outer circle,

$$R=a_1.$$

As an example of the application of this method, let us take the case of a coil of wire, in which the wires are arranged so that the transverse section of the coil exhibits the sections of the wires arranged in square order, the distance between two consecutive wires being D, and the diameter of each wire d.

Let the whole section of the coil be of dimensions which are small compared with the radius of curvature of the wires, and let the geometrical mean distance of the section from itself be R.



Let it be required to find the coefficient of induction of this coil on itself, the number of windings being n.

1st. If we begin by supposing that the wires fill up the whole section of the coil, without any interval of insulating matter, then if M is the coefficient of induction of a linear circuit of the same shape as the coil on a similar parallel circuit at a distance R, the coefficient of induction of the coil on itself will be

$$n^2 M.$$

2nd. The current, however, is not uniformly distributed over the section. It is confined to the wires. Now the coefficient of self-induction of a unit of length of a conductor is

$$C - 2 \log R,$$

where C is a constant depending on the form of the axis of the conductor, and R is the mean geometric distance of the section from itself.

Now for a square of side D,

$$\log R_1 = \log D + \tfrac{1}{3} \log 2 + \frac{\pi}{3} - \tfrac{25}{12},$$

and for a circle of diameter d

$$\log R_2 = \log d - \log 2 - \tfrac{1}{4}.$$

Hence $$\log \frac{R_1}{R_2} = \log \frac{D}{d} + \tfrac{4}{3} \log 2 + \frac{\pi}{3} - \tfrac{11}{6},$$

and the coefficient of self-induction of the cylindric wire exceeds that of the square wire by

$$2 \left\{\log \frac{D}{d} + 0{\cdot}1380606\right\}$$

per unit of length.

3rd. We must also compare the mutual induction between the cylindric wire and the other cylindric wires next it with that between the square wire and the neighbouring square wires. The geometric mean distance of two squares side by side is to the distance of their centres of gravity as 0·99401 is to unity.

The geometric mean distance of two squares placed corner to corner is to the distance between their centres of gravity as 1·0011 is to unity.

Hence the correction for the eight wires nearest to the wire considered is

$$-2 \times (0{\cdot}01971).$$

The correction for the wires at a greater distance is less than one-thousandth per unit of length.

The total self-induction of the coil is therefore

$$n^2 M + 2l \left\{\log \frac{D}{d} + 0{\cdot}11835\right\},$$

where n is the number of windings, and l the length of wire.

For a circular coil of radius $= a$,

$$M = 4\pi a\,(\log 8a - \log R - 2),$$

where R is the geometrical mean distance of the section of the coil from itself.