

# Practical continuous functions for the internal impedance of solid cylindrical conductors

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## Abstract

Methods for calculating the internal impedance of round wires are investigated. 'Exact' calculation using Kelvin Bessel functions runs into difficulties at radio frequencies due to rounding errors in computer floating-point arithmetic. Specialist techniques (such as the use of high-precision BCD arithmetic) could be used to circumvent this problem; but for general modelling, the use of approximations is common practice. The traditional 'thick-conductor approximation' for AC resistance is inaccurate and has an incorrect lower limit. An improved derivation with allowance for surface curvature gives rise to an asymptotic form that converges with the Bessel calculation for only moderately large arguments. This is modified to give a bridging polynomial, that is used in conjunction with an optimised Kelvin function algorithm to give a calculation routine with a maximum error of  $< 0.01$  ppM (assuming double-precision arithmetic) and no upper frequency limit. A bridging polynomial is also used for the internal inductance case and gives the same overall accuracy.

Inaccuracy and range restrictions can also be avoided without the need for complicated computer programs. A generalised method for producing continuous doubly asymptotically-correct approximations (ACAs) accurate to within a few percent is demonstrated. Suitable choice of ACA allows further correction using modified Lorentzian (ML) functions, leading to a family of compact formulae. The best of these for AC resistance (Rac-TEDML) is accurate to within  $\pm 0.09\%$ ; and the best for internal inductance (Li-PACAML) is accurate to within  $\pm 0.02\%$ .

Keywords:

skin effect, internal impedance, internal inductance, Kelvin Bessel functions.

## Acknowledgements

I would like to thank Robert Weaver<sup>3</sup> for helpful discussions and considerable assistance. Program routines described here have been developed from Bob's code examples and from his comments on how to minimise error and computation time when calculating infinite series. The empirical formulae given here have been optimised using his Nelder-Mead 'Non-Linear Solver' multi-parameter fitting routine.

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## 1. Introduction

The internal impedance of a conductor is the phasor combination of AC resistance and internal inductance. It can be written:

$$\mathbf{Z}_i = R_{ac} + j2\pi f L_i$$

The AC resistance,  $R_{ac}$ , is the same as the DC resistance at low frequencies, but increases with frequency as a consequence of the skin effect. The internal inductance,  $L_i$ , which is associated with the magnetic field within the conductor, is  $\mu/(8\pi)$  per metre at low frequencies for all isolated wires and conductors of circular cross-section (50 nH/m for non-ferromagnetic materials), but diminishes with frequency as a consequence of the skin effect. Hence internal impedance, when furnished with expressions for  $R_{ac}$  and  $L_i$  that are suitable for practical calculation, accounts for the skin effect in the manner required for the purpose of RF circuit modelling.

$R_{ac}$ , of course, must be quantified when estimating the efficiency of antennas, transmission lines, coils, transformers, and so on. Its significance is therefore generally appreciated, and its calculation is a matter of some importance.  $L_i$  is more likely to be neglected, but its inclusion can usefully improve the accuracy of inductance calculations; especially when relatively thin wire is involved. It is therefore unfortunate that the exact calculation of  $R_{ac}$  and  $L_i$  is a matter of some difficulty; a situation that leads to the use of approximations that can easily become inaccurate or invalid without vigilance on the part of the investigator.

The exact calculation of the internal impedance of a solid cylindrical conductor or 'round wire' involves Bessel functions of the first kind, of zero order, with complex arguments (and derivatives thereof). This type of problem can be separated into real and imaginary parts using the Kelvin Bessel functions,  $ber$ ,  $bei$ ,  $ber'$  and  $bei'$ . The Kelvin functions are computed from infinite series expansions of alternating sign, and higher terms involve large powers of the argument divided by large factorials; small increments to the returned value being obtained by subtraction of one very large number from another. This, unfortunately, is the kind of procedure that falls foul of the range limitations inherent in standard computer arithmetic. For example, in experiments with single-line formulae entered into the Open Office Calc<sup>4</sup> spreadsheet program, it was found that no further improvement could be obtained, indeed a degradation of accuracy occurred, for Kelvin function series of more than 9 terms. This had the effect of limiting the calculation of the internal impedance of copper wires up to 30 MHz to a maximum wire diameter of about 0.25 mm.

Using arbitrarily truncated series formulae is, of course, an unsatisfactory approach to such a tricky computational problem. Far better results can be obtained by use of incremental methods and recursion relations. This issue was investigated by Robert Weaver, and led to his writing of a set of double-precision Basic programs implemented as Open Office macros<sup>5</sup>. Bob's optimised calculation routines extend the Bessel function argument range considerably, and were used as a basis for this work; but they can still become inaccurate with parameters likely to be encountered when solving HF and VHF radio problems.

The troublesome nature of Bessel functions of large argument has led to the development of polynomial approximations (discussed later). These give accuracies of the order of 0.1 parts-per-million (ppM); but involve large numbers of empirical parameters. They therefore constitute an over-complicated approach in many applications, particularly those for which accuracy can be sensibly traded for simplicity in view of the uncertainties of the input parameters.

In order to push the calculation of internal impedance to high-frequencies (or large diameters) without special programming or undue mathematical complexity; it is common practice to use the so-called 'thick conductor' approximations. The basic thick conductor approximation for internal inductance is easily dispensed, since it amounts to the assumption that there isn't any. The

<sup>4</sup> Available for free download from [openoffice.org](http://openoffice.org). Required for viewing and editing the accompanying spreadsheet files and macros.

<sup>5</sup> **Practical considerations in the calculation of Kelvin Bessel functions and complete elliptic integrals  $K$  and  $E$ .** Robert S Weaver, Oct. 2009. <http://electronbunker.ca/DLpublic/KelvinEllipticCalcs.pdf> (accessed 2016-04-27)

approximation for AC resistance however, is based on the assumption that surface curvature is small and thickness large, so that the solution of Maxwell's equations for field penetration into a plane conductor of infinite thickness prevails. As we will see, this approximation can give excellent results when subjected to a simple correction; but the most widely accepted formula, which hails from a time when radio components were specified to a tolerance of 20%, is still inaccurate at hundreds of MHz. That the traditional crude approximation is thought by some to be exact is a caveat for those who use computer programs of unknown provenance; but there is still an issue even when a better derivation is used. The problem is that thick conductor approximations behave incorrectly in the limits of low frequency or small wire diameter. Hence they can give wildly inaccurate results in the hands of the inexperienced or unsuspecting user.

Even for those who use the available computation methods with confidence, the lack of closed-form continuous formulae for internal impedance is an inconvenience and encourages mistakes. Hence the work to be described below, which was prompted by a desire to model the RF dispersion behaviour (variation of inductance with frequency) of solenoid inductors wound with thin wire; the required accuracy being anything better than one part in 1000. Since the test coils had a low-frequency internal inductance component of about 1% of the total, the original intention was merely to obtain a computationally straightforward doubly-asymptotic expression accurate to better than about 5%. That target was comfortably exceeded; the outcome being a family of approximations, using only standard library functions, the best of which are accurate to within 0.1%. These formulae, being both compact and efficient, should be useful to others; and a report on their origins is necessary as background to other work.

Attendant on the development of the formulae; it has, of course, been necessary to obtain accurate reference data for the adjustment of empirical coefficients. Hence this work also includes an investigation of the convergence between high-frequency asymptotic forms of the expressions for AC resistance and internal inductance and the corresponding Bessel function combination. This leads to calculation routines that branch contiguously to an asymptotic form at the point where the Bessel calculation starts to become inaccurate. These are given below as single-argument spreadsheet macro functions, accurate to better than 0.01ppM, which can be adopted in preference to the formulae in the event that program execution speed is not critical.

## 2. Choice of fitting functions

The method by which the approximations were obtained will be described because it may have relevance in other situations. Particularly, the use of modified Lorentzian functions appears to have an application in correcting approximate formulae in cases where the error is always positive or always negative. Such is the case, or can be forced to be the case, for a range of problems when a doubly-asymptotic approximation is chosen as the starting point.

In the process of developing formulae for AC resistance, the first concern was to find a correction function linking the DC and the high-frequency limiting cases. This problem was solved by means of a first-order current-density magnitude model for skin effect, which will be referred-to here as 'truncated exponential decay' (TED). The idea is that, since skin depth is derived from the area under an exponential decay curve extending from the surface to infinite depth, then for conductors of finite thickness, a modified skin depth parameter can be obtained by determining the area under a truncated decay curve. It was found that by truncating the curve at the wire axis (maximum depth equal to the radius), the new skin depth parameter was forced equal to the radius at low frequencies, and the expression for resistance converged with the chosen thick-conductor approximation at high frequencies, i.e., a doubly asymptotically-correct approximation was obtained.

On producing graphs of the difference between the TED and the 'exact' model, it was noted that the error was concentrated in a single peak of highly skewed appearance when plotted on either linear or logarithmic frequency scales. When plotted on a scale of the fourth-root of frequency however, the error distribution took on an approximately symmetric appearance of roughly Gaussian profile. This was compared against a true Gaussian using initially guessed parameters for centre frequency and width (the height already being known). Then, by varying the independent input parameters;  $d$  (wire diameter) and  $\rho/\mu$  (resistivity/permeability), expressions for the centre frequency and the width were found. This first attempt produced a formula accurate to within 0.7%.

It was noted that a good deal of the error in the TED-G model was due to residual asymmetry. The first issue here is that the actual error function must go to zero as frequency goes to zero, whereas the Gaussian does not. This problem can be resolved by modifying the independent variable ( $x$ ). Specifically; a true Gaussian, normalised to have a maximum height of 1, has the form:

$$y = \exp[-w(x - \bar{x})^2]$$

where  $w$  is the width-determining parameter and  $\bar{x}$  is the value of  $x$  at the peak. A modified form that is forced to zero when  $x=0$  is:

$$y = \exp\left[-w\left(\frac{x}{\bar{x}} - \frac{\bar{x}}{x}\right)^2\right]$$

Using this form did indeed improve the fit, but the asymmetry introduced by correcting the zero-frequency limit was not sufficient to match the skew of the actual error curve. An arbitrary amount of skew can however be introduced by multiplying the width parameter by a dimensionless normalised function of  $x$ , i.e., the desired effect can be obtained by allowing the 'width' to vary with  $x$ .

A Gaussian correction function was not used in the final formulae however, because a modified Lorentzian was always found to give a better fit. The Lorentzian also has the advantage that it can provide an extra degree of freedom if necessary, due to the fact that the width of a normalised lineshape function can be altered by raising it to an arbitrary power. In the case of a Gaussian, raising to a power has exactly the same effect as changing the width parameter. In the case of a Lorentzian however, raising to a power has more effect in the wings of the distribution than it has at the half-height. Hence, using a modified Lorentzian, it becomes possible to tailor the wings and the half-width independently.

A true Lorentzian, normalised to have a maximum height of 1, has the form:

$$y = \frac{1}{1 + w(x - \bar{x})^2}$$

This curve is symmetric about  $\bar{x}$ , as is the Gaussian; and can be skewed to establish the limiting condition that  $y \rightarrow 0$  as  $x \rightarrow 0$  in much the same way:

$$y = \frac{1}{1 + w\left(\frac{x}{\bar{x}} - \frac{\bar{x}}{x}\right)^2}$$

This form, incidentally, is known to engineers as the electrical resonance curve. Now, raising the

whole thing to an arbitrary power  $p$ , we obtain a function with two width parameters of differing effect:

$$y = \frac{1}{\left[1 + w \left(\frac{x}{\bar{x}} - \frac{\bar{x}}{x}\right)^2\right]^p}$$

Now we need to introduce variable skew, but before doing so, a discussion of the argument  $x$  is in order. It was mentioned earlier, that  $x$  was found to be proportional to the fourth-root of frequency. It was also found, for the AC resistance problem, that the peak in the difference curve, for any values of the input parameters, always occurred when

$$\frac{d}{\delta_i} = \pi$$

where  $d$  is the wire diameter and  $\delta_i$  is the thick-conductor skin depth, defined as:

$$\delta_i = \sqrt{\frac{\rho}{\pi f \mu}}$$

Hence,  $d/\delta_i$  is proportional to the square-root of frequency, and so:

$$x + \sqrt{\frac{d}{\delta_i}} \quad \text{and} \quad \bar{x} = \sqrt{\pi}$$

Hence:

$$\frac{x}{\bar{x}} = \sqrt{\frac{d}{\delta_i \pi}}$$

Some multiple or function of  $d/\delta_i$  is always the argument (input value) for the internal impedance problem; i.e., although there are several input parameters, internal impedance is actually a function of a single argument. Hence we have a single argument for the correction function, normalised (brought to unity when  $x = \bar{x}$ ) by inclusion of the factor  $1/\sqrt{\pi}$ .

Such a normalised function is also what is required in order to skew the curve, i.e., we need to multiply the width parameter  $w$  by  $x/\bar{x}$  raised to some arbitrary power. Thus:

$$y = \frac{1}{\left[1 + w \left(\frac{x}{\bar{x}}\right)^a \left(\frac{x}{\bar{x}} - \frac{\bar{x}}{x}\right)^2\right]^p}$$

If that is the case however, then we can just as well multiply the skewing function into the bracket to its right:

$$y = \frac{1}{\left[1 + w \left[\left(\frac{x}{\bar{x}}\right)^b \frac{x}{\bar{x}} - \left(\frac{x}{\bar{x}}\right)^b \frac{\bar{x}}{x}\right]^2\right]^p}$$

(where  $b = \sqrt{a}$ ). Combining the powers we get:

$$y = \frac{1}{\left[ 1 + w \left[ \left( \frac{x}{\bar{x}} \right)^{b+1} - \left( \frac{x}{\bar{x}} \right)^{b-1} \right]^2 \right]^p}$$

Finally we note that even greater versatility can be obtained, without affecting the normalisation, by allowing the powers to which the two  $x/\bar{x}$  terms are raised to take on arbitrary values. Also, we introduce a parameter  $h$ , which sets the height at the peak. Thus:

|   |                           |
|---|---------------------------|
| $y = \frac{h}{\left[ 1 + w \left[ \left( \frac{x}{\bar{x}} \right)^{p_1} - \left( \frac{x}{\bar{x}} \right)^{p_2} \right]^2 \right]^p}$ | ML<br>Modified Lorentzian |
|---|---------------------------|

This function, in addition to its quasi-independent adjustments for skirt-width and half-width, allows the shapes of the curves on either side of the peak to be adjusted independently. It will be referred to in this document as a modified Lorentzian (ML) function.

Notice that the relationship between the natural argument for the problem (in this case some multiple of  $d/\delta_i$ ) and the ML argument ( $x/\bar{x}$ ) is in principle arbitrary. In the case where transformation from one to the other is simply a matter of multiplying by a constant and raising to a power however, then the power can be absorbed into the parameters  $p_1$  and  $p_2$ , and the constant can be absorbed into  $\bar{x}$ . Hence, in the internal impedance case; there is no need for transformation prior to fitting, because the goodness of the fitting process will not be affected if physically significant parameters are extracted by transformation after the event.

The TED-ML approach can be used to produce good approximations for AC resistance and internal inductance, but modifications that give additional degrees of freedom are possible. One method involves altering the power or order of the function used to achieve the initial asymptotically-correct approximation (ACA). Since, after modification, the function no longer strictly represents truncated exponential decay, it will be referred to here as ACAn, where  $n$  is the order (and ACA1 is the same as TED in the AC resistance case). By varying  $n$  (which need not be an integer), it is possible to find ACAs that are accurate to within a few percent without further correction. The order also varies the shape of the error function, from bell-shaped curves of positive or negative height, to S-shaped curves in-between. Adjusting  $n$  to obtain a substantially all-positive or all-negative error curve is, of course, a prerequisite for ML correction. In the AC resistance case, the best order will be shown to be about 1, giving a maximum error of  $\pm 0.09\%$  after ML correction and confirming the TED hypothesis as a physically reasonable first-order model. In the internal inductance case, the best order turns out to be 3.74, giving a maximum error of  $\pm 0.034\%$  after ML correction, but implying that the fitting parameters have no direct physical significance. A possible disadvantage of ACAn as a general-purpose fitting technique is that, when  $n$  is large, it can exhibit rounding-error when the input argument is very small. A solution to this problem, as will be demonstrated, is to limit the order by incorporating a simple polynomial into the transformation that converts the initial approximation from singly-asymptotic to doubly-asymptotic form. The polynomial ACA approach with ML correction leads to an expression for internal inductance accurate to  $\pm 0.016\%$ .

### 3. Exact expression for internal impedance

The solution of Maxwell's equations for the internal impedance of a round wire of high conductivity and uniform composition is given by various authors. See for example Ramo et al.<sup>6</sup> or McLachlan<sup>7</sup>. Expressed using the Kelvin Bessel functions it is:

$$\mathbf{Z}_{\text{int}} = R_{\text{ac}} + \mathbf{j} 2\pi f L_i = \frac{\mathbf{j} \ell R_s}{\sqrt{2} \pi r} \left[ \frac{\text{Ber}(q) + \mathbf{j} \text{Bei}(q)}{\text{Ber}'(q) + \mathbf{j} \text{Bei}'(q)} \right] \quad [ \Omega / \text{m} ] \quad (3.1)$$

Where  $\ell$  is the conductor length (i.e. the length of the wire),  $r = d/2$  is the wire radius, and  $R_s$  is the surface resistivity defined as:

$$R_s = \sqrt{\pi f \mu \rho} \quad [ \Omega ]$$

The argument of the Kelvin functions,  $q$ , is defined as:

$$q \equiv \frac{\sqrt{2} r}{\delta_i} = \frac{d}{\sqrt{2} \delta_i}$$

where  $\delta_i$  is the 'good conductor' skin depth (i.e., the depth at which the current density has dropped to  $1/e$  of its surface value) in the limit of infinite conductor thickness and zero surface curvature.  $\delta_i$ , as mentioned previously, is given by the expression:

$$\delta_i = \sqrt{\frac{\rho}{\pi f \mu}}$$

$\rho$  being the volume resistivity, and  $\mu$  the absolute magnetic permeability of the conductor. Note that, in this article,  $\mu$  is given without a subscript; but, in a more general context, it will require a subscript to distinguish it from the permeability of the medium outside the wire.

The expression for internal impedance is put into  $a + \mathbf{j}b$  form by multiplying the external  $\mathbf{j}$  into the square bracket (noting that  $\mathbf{j}^2 = -1$ ), multiplying numerator and denominator by the complex conjugate of the denominator, and then multiplying out and rearranging. This gives:

$$\mathbf{Z}_{\text{int}} = \frac{\ell R_s}{\sqrt{2} \pi r} \left[ \frac{\text{Ber}(q) \text{Bei}'(q) - \text{Bei}(q) \text{Ber}'(q) + \mathbf{j} [\text{Ber}(q) \text{Ber}'(q) + \text{Bei}(q) \text{Bei}'(q)]}{\text{Ber}'(q)^2 + \text{Bei}'(q)^2} \right] \quad (3.2)$$

$R_{\text{ac}}$  is then the real part, and the imaginary part is the internal reactance. Note that it is implicit in equation (3.1) that internal reactance is purely inductive, because a good conductor cannot sustain an electric field. That is not strictly true, because metal wires have finite resistance<sup>8</sup>; but it is true to an extremely good approximation, such that there is little sleight of hand in claiming the expression to be exact in practice.  $L_i$  is thus the imaginary part of (3.2) divided by  $2\pi f$ .

6 **Fields and Waves in Communication Electronics**, 3rd Edition, Simon Ramo, John R. Whinnery, Theodore Van Duzer, 3rd edition. Publ. John Wiley & Sons Inc. 1994. ISBN 0-471-58551-3.

3.16 (p149-153): Penetration of Electromagnetic Fields into a Good Conductor (skin effect).

4.4 (p180-182): Skin effect in practical conductors. 4.5 (p182-186): Impedance of round wires.

7 **Bessel Functions for Engineers**. N W McLachlan. 2nd edition. Oxford, Clarendon Press 1955.

8.10 (p153-159) Current density in wire carrying alternating current.

8 **Oliver Heaviside**, Paul J Nahin. 2nd edition (paperback). John Hopkins University Press 2002. ISBN 0-8018-6909-9. Heaviside's general formula for skin effect (i.e., including poorly conducting materials and hence involving permittivity as well as permeability) is given on pages 175-176, with citation of the original 1888 reference.



The Kelvin functions, Ber and Bei (Bessel real and Bessel imaginary) can be evaluated using the series expansions:

$$\text{Ber}(q) = 1 - \frac{(q/2)^4}{2!^2} + \frac{(q/2)^8}{4!^2} - \frac{(q/2)^{12}}{6!^2} + \frac{(q/2)^{16}}{8!^2} - \frac{(q/2)^{20}}{10!^2} + \dots$$

$$\text{Bei}(q) = (q/2)^2 - \frac{(q/2)^6}{3!^2} + \frac{(q/2)^{10}}{5!^2} - \frac{(q/2)^{14}}{7!^2} + \frac{(q/2)^{18}}{9!^2} - \frac{(q/2)^{22}}{11!^2} + \dots$$

The functions Ber' and Bei' are the first derivatives of Ber and Bei and can be obtained by differentiating the series term by term. Thus:

$$\text{Ber}'(q) = -\frac{4q^3}{2^4 2!^2} + \frac{8q^7}{2^8 4!^2} - \frac{12q^{11}}{2^{12} 6!^2} + \frac{16q^{15}}{2^{16} 8!^2} - \frac{20q^{19}}{2^{20} 10!^2} + \dots$$

$$\text{Bei}'(q) = \frac{q}{2} - \frac{6q^5}{2^6 3!^2} + \frac{10q^9}{2^{10} 5!^2} - \frac{14q^{13}}{2^{14} 7!^2} + \frac{18q^{17}}{2^{18} 9!^2} - \frac{22q^{21}}{2^{22} 11!^2} + \dots$$

The Kelvin functions converge rapidly for arguments (values of  $q$ ) less than 1, and in that case may be evaluated to good accuracy using only the first three terms. For larger arguments however, convergence is slow, more terms are needed, and some method for dealing with floating-point rounding error is required.

For a preliminary version of this study, the functions were evaluated using 9-term expansions. These were entered into a spreadsheet as single-line formulae, using the native exponentiation operator and factorial function. Transcription was verified by comparing the returned values with McLachlan's tables<sup>9</sup> (which go up to  $q=10$  and can be reproduced using 8-term expansions). As mentioned in the introduction; due to rounding errors, no improvement could be obtained by going beyond 9 terms; and at least one of the functions became seriously unstable for arguments greater than 15. Fortunately, the internal impedance calculation is convergent with the asymptotic forms used somewhat before  $q$  reaches 15; and so in a reversal of roles, the approximation served to monitor the health of the 'exact' result.

The truncated series formulae now only appear in this work as a demonstration of the limitations of that approach. They have been superseded by carefully optimised program routines, which are based on code developed by Bob Weaver and kindly made available. Bob's algorithms are accurate within a few parts-per-million for arguments up to 85, and still give acceptable results for arguments up to about 95. Also, as we will see in the next few sections; the four functions can be implemented in a single program loop, and they can be supplemented by asymptotic forms to give high-precision internal impedance functions that are effectively free from argument-range restrictions. Note that, although the method is not used here, internal impedance can also be determined by representing it as an infinite (but truncatable) set of parallel RL elements<sup>10</sup>.

<sup>9</sup> **Bessel Functions for Engineers**. N W McLachlan. 2nd edition. Oxford, Clarendon Press 1955. p222-223

<sup>10</sup> **Frequency-dependent skin-effect formulation for resistance and internal inductance of a solid cylindrical conductor**. O M O Gatous and J Pissolato. IEE Proc.- Microwave Antennas & Propagation. 151(3), Jun. 2004. p212-216. Also given as a conference paper: **A New Formulation for Skin-effect Resistance and Internal Inductance Frequency-Dependent of a Solid Cylindrical Conductor**. O M O Gatous and J Pissolato Filho. IEEE/PES Transmission & Distribution Conf. & Expo. Latin America, 2004. p919-924.

## AC Resistance

### 4. AC resistance factor

The resistance of a conductor at low frequencies is given by:

$$R_{dc} = \frac{\rho \ell}{A} \quad [ \Omega ]$$

where  $\ell$  is the length,  $A$  is the cross-sectional area, and  $\rho$  is the volume resistivity of the conduction path. For a conductor of circular cross-section, this becomes:

$$R_{dc} = \frac{\rho \ell}{\pi r^2} \quad [ \Omega ]$$

where  $r$  is the cylinder radius. This relationship can also be expressed as resistance per unit length:

$$\frac{R_{dc}}{\ell} = \frac{\rho}{\pi r^2} \quad [ \Omega / m ]$$

We can envisage the skin effect as reducing the area available for conduction as the frequency increases. Hence we can define AC resistance as:

$$R_{ac} = \frac{\rho \ell}{A_{eff}}$$

where the effective area,  $A_{eff}$ , is a function of frequency.

One of the easiest ways in which to include AC resistance in electrical models is to capture it in a dimensionless AC resistance factor, or 'skin-effect factor'. Following Medhurst<sup>11</sup>, the symbol  $\Xi$  (Greek Xi) will be used here. Thus:

$$R_{ac} = R_{dc} \Xi$$

$$\Xi = \frac{R_{ac}}{R_{dc}} = \frac{\rho / A_{eff}}{\rho / A}$$

Hence, by definition:

$$\Xi \equiv \frac{R_{ac}}{R_{dc}} \equiv \frac{A}{A_{eff}} \quad \text{AC resistance factor. Skin-effect factor.}$$

By obtaining expressions for  $A_{eff}$  and  $\Xi$ , various formulae for AC resistance can be compared in a consistent manner.

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11 "H. F. Resistance and Self-Capacitance of Single-Layer Solenoids", R G Medhurst (GEC Research Labs.), Wireless Engineer, Feb. 1947 p35-43, Mar. 1947 p80-92.

## 5. Exact expression for AC resistance

To obtain an 'exact' expression for the AC resistance factor, we start by identifying  $R_{ac}$  as the real part of equation (3.2). Thus:

$$\frac{R_{ac}}{\ell} = \frac{\rho}{A_{eff}} = \frac{R_s}{\sqrt{2}\pi r} \left[ \frac{\text{Ber}(q)\text{Bei}'(q) - \text{Bei}(q)\text{Ber}'(q)}{\text{Ber}'(q)^2 + \text{Bei}'(q)^2} \right] \quad [ \Omega / m ]$$

Where  $R_s = \sqrt{\pi f \mu \rho}$  ,  $q = \frac{\sqrt{2}r}{\delta_i}$  and  $\delta_i = \sqrt{\frac{\rho}{\pi f \mu}}$  .

Substituting for  $R_s$  gives:

$$\frac{R_{ac}}{\ell} = \frac{\rho}{A_{eff}} = \frac{\sqrt{\pi f \mu \rho}}{\sqrt{2}\pi r} \cdot \left[ \frac{\text{Ber}(q)\text{Bei}'(q) - \text{Bei}(q)\text{Ber}'(q)}{\text{Ber}'(q)^2 + \text{Bei}'(q)^2} \right]$$

Now, by multiplying top and bottom by  $\pi r$ , and collecting parameters to separate  $1/\delta_i$  we get:

$$\frac{R_{ac}}{\ell} = \frac{\rho}{\pi r^2} \frac{r}{\sqrt{2}\delta_i} \left[ \frac{\text{Ber}(q)\text{Bei}'(q) - \text{Bei}(q)\text{Ber}'(q)}{\text{Ber}'(q)^2 + \text{Bei}'(q)^2} \right]$$

$\rho/\pi r^2$  is of course  $R_{dc}/\ell$  , and so we get:

$$\Xi = \frac{R_{ac}}{R_{dc}} = \frac{r}{\sqrt{2}\delta_i} \left[ \frac{\text{Ber}(q)\text{Bei}'(q) - \text{Bei}(q)\text{Ber}'(q)}{\text{Ber}'(q)^2 + \text{Bei}'(q)^2} \right] \quad \text{'exactly'} \quad (5.1)$$

Alternatively, we can create an expression for  $A_{eff}$ . Before doing that however, it will be convenient to make the following definition:

$$U(q) = \left[ \frac{\text{Ber}(q)\text{Bei}'(q) - \text{Bei}(q)\text{Ber}'(q)}{\text{Ber}'(q)^2 + \text{Bei}'(q)^2} \right]$$

which allows us to write:

$$\Xi = \frac{A}{A_{eff}} = \frac{r U(q)}{\sqrt{(2)}\delta_i} = \frac{q}{2} U(q) \quad , \quad \text{where} \quad q = \frac{\sqrt{2}r}{\delta_i} \quad \text{and} \quad \delta_i = \sqrt{\frac{\rho}{\pi f \mu}}$$

Hence, noting that  $A = \pi r^2$  and  $r = d/2$  :

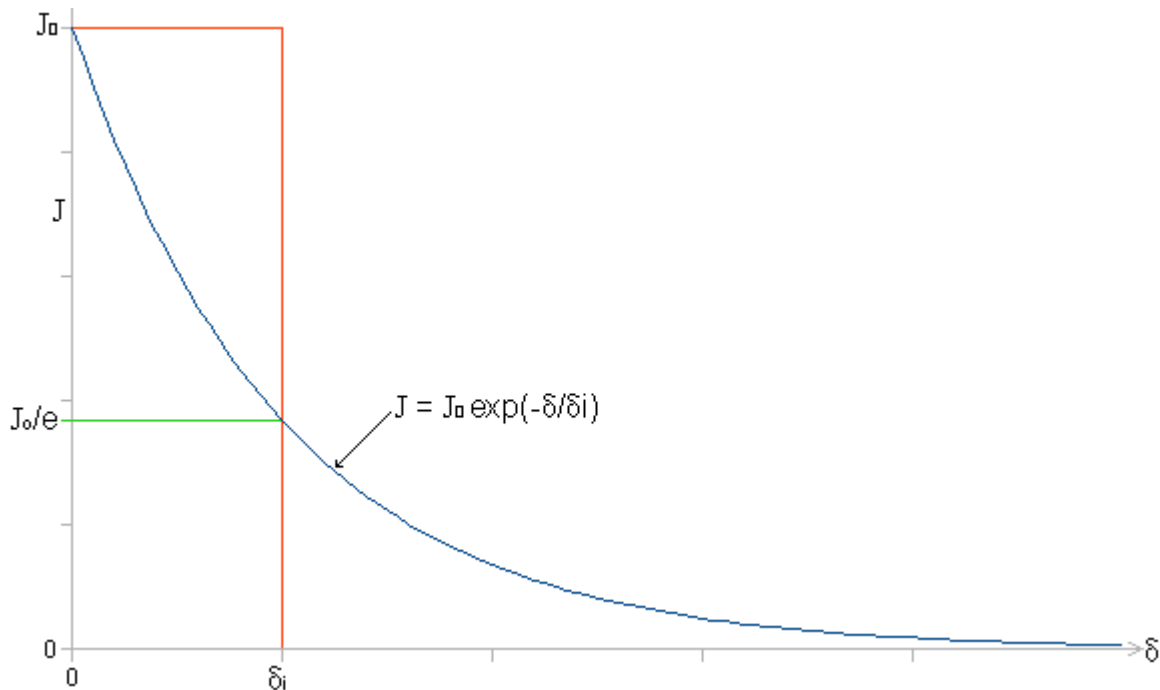
$$A_{eff} = \frac{\sqrt{2}\pi r \delta_i}{U(q)} = \frac{\pi d \delta_i}{\sqrt{2} U(q)} \quad [ m^2 ] \quad \text{'exactly'}$$

(where 'exactly' should be taken to mean 'provided that  $q$  is not so large as to cause computational errors'). Note that when  $f \rightarrow 0$  ,  $\delta_i \rightarrow \infty$  and so  $q \rightarrow 0$  . When  $q = 0$  ,  $\text{Ber}(q) = 1$  and  $\text{Bei}(q) = \text{Ber}'(q) = \text{Bei}'(q) = 0$  . Hence  $\Xi \rightarrow 0 \times 0/0$  and is indeterminate. This is because zero frequency is, strictly, physically impossible. We can resolve this limit without resorting to l'Hôpital's rule however, by noting that when  $f \rightarrow 0$  ,  $R_{ac} \rightarrow R_{dc}$  . Hence  $\Xi_0 = 1$  . Therefore:

as  $q \rightarrow 0$ ,  $U(q) \rightarrow 2/q$

## 6. Thick conductor approximations, good and bad

In a conductor of infinite thickness and zero curvature, the magnitude of the conduction current density decays exponentially with depth. The skin depth is the hypothetical depth to which the current would penetrate if it flowed uniformly at shallow depths and then ceased abruptly. Its derivation is illustrated below:



$J_0$  is the magnitude of the current density at the surface of the conductor, and the skin depth  $\delta_i$  is defined such that the area  $J_0 \delta_i$  is the same as the area under the decay curve:

$$J = J_0 \exp\left(-\delta \sqrt{\frac{\pi f \mu}{\rho}}\right)$$

Where  $\delta$  is the distance from the surface. The area under the curve is, of course, given by the integral; and the standard integral required in this case is:

$$\int \exp(ax) dx = \frac{1}{a} \exp(ax) + c$$

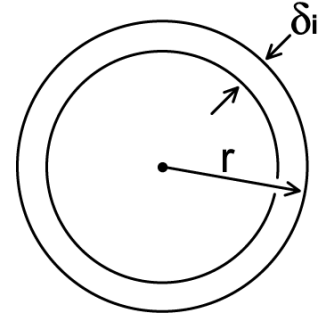
Thus:

$$\int_0^{\infty} J_0 \exp\left(-\delta \sqrt{\frac{\pi f \mu}{\rho}}\right) d\delta = J_0 \left[ -\sqrt{\frac{\rho}{\pi f \mu}} \right] [\exp(-\infty) - \exp(0)] = -J_0 \sqrt{\frac{\rho}{\pi f \mu}} [0 - 1]$$

Hence the area  $J_0 \sqrt{\frac{\rho}{\pi f \mu}}$  is the same as  $J_0 \delta_i$ ; and  $\delta_i$ , the depth at which the current has fallen to  $1/e = 0.368$  of its surface value is thus:

$$\delta_i = \sqrt{\frac{\rho}{\pi f \mu}}$$

The virtue of the abrupt cutoff model is that, presuming that the thickness of the conductor is much greater than  $\delta_i$ , it allows the effective cross-sectional area to be calculated easily. In the case of a cylindrical conductor, subject to the assumption that the radius is sufficiently large to permit the use of the plane-conductor solution for field penetration; the effective area is simply the total area minus the area of the 'non-conducting' inner region. Referring to the diagram on the right, the radius of the inner region is  $r - \delta_i$ , and so the effective area is:



$$A_{\text{eff}} = \pi r^2 - \pi (r - \delta_i)^2 = \pi r^2 - \pi (r^2 - 2r\delta_i + \delta_i^2)$$

Thus:

$$A_{\text{eff}} = \pi (2r\delta_i - \delta_i^2) = \pi (d\delta_i - \delta_i^2) \quad [m^2] \quad r \gg \delta_i \quad (6.1)$$

$$\Xi = \frac{A}{A_{\text{eff}}} = \frac{R_{\text{ac}}}{R_{\text{dc}}} = \frac{r^2}{2r\delta_i - \delta_i^2} \quad \text{Thick-conductor approximation (good)} \quad (6.2)$$

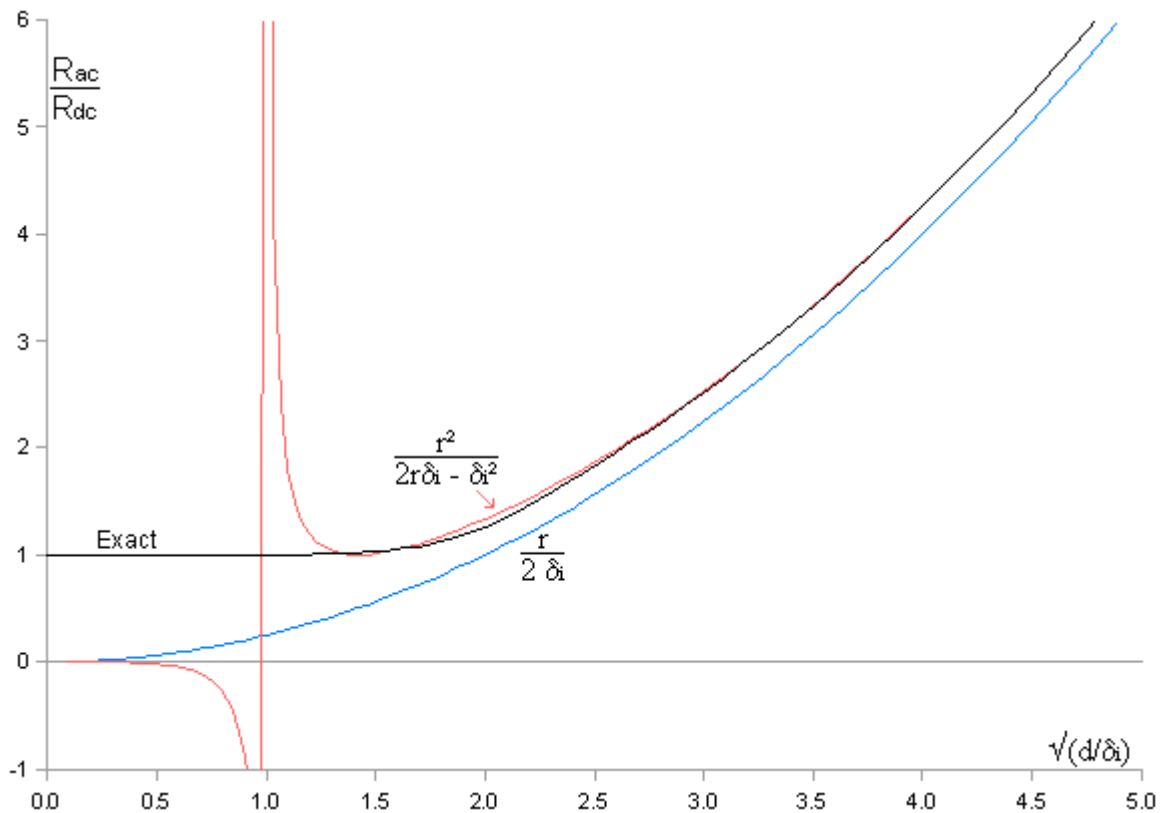
This expression can be used as written or, as pointed out by Wheeler<sup>12</sup>, can be expanded as a series and truncated. Further approximation is also possible by observing that, when  $\delta_i$  is small relative to  $r$ ,  $\delta_i^2$  will be extremely small. Hence it is arguable that the  $\delta_i^2$  term can be deleted, giving:

$$\Xi = \frac{A}{A_{\text{eff}}} = \frac{R_{\text{ac}}}{R_{\text{dc}}} = \frac{r}{2\delta_i} \quad \text{Thick-conductor approximation (bad)} \quad (6.3)$$

$$A_{\text{eff}} = 2\pi r \delta_i = \pi d \delta_i \quad [m^2] \quad \frac{r}{\delta_i} \rightarrow \infty \quad (6.4)$$

Notice that  $2\pi r \delta_i$  is the area of a rectangle of length equal to the circumference and width  $\delta_i$ . Thus it is possible to deduce this approximation without the intermediate step of deriving (6.1). This might explain why (6.4) finds widespread use; even though (6.1) is vastly superior and only a little more complicated.

The two thick conductor approximations are compared with the exact expression (5.1) in the following graph; which shows  $R_{\text{ac}}/R_{\text{dc}}$  vs.  $\sqrt{d/\delta_i}$ , the latter being proportional to the fourth-root of frequency.



The 'good' approximation goes to infinity when  $d = 2r = \delta_i$ , i.e., when  $\sqrt{d/\delta_i} = 1$ ; but then converges rapidly with the exact calculation as  $d/\delta_i$  increases. If the formula is used for  $\sqrt{d/\delta_i} > \sqrt{2}$ , i.e.,  $r > \delta_i$ , the maximum error is 5.5%. As a practical example; for a 0.5 mm diameter copper wire,  $r = \delta_i$  at 700 kHz.

Convergence with the exact calculation within 0.1% occurs when  $d/\delta_i \geq 18$ . Recall that the argument for the Kelvin functions is  $q = d / \delta_i \sqrt{2}$ , and so the corresponding value for  $q$  is 12.7. It was mentioned earlier, that single-line 9-term formulae become inaccurate when  $q > 15$ ; but there is a useful overlap between the the two calculation methods. Thus we see that reasonably accurate mathematical modelling can be accomplished even when the Bessel functions are coded simply, by the expedient of handing over to the good thick-conductor approximation at the point where the 'exact' calculation runs into difficulties.

It is perhaps unfair to refer to (6.3) as the 'bad' thick-conductor approximation, because it was used in some widely cited studies in the early part of the 20<sup>th</sup> Century. Those studies however, were carried out before engineers and scientists had access to electronic computers, and the correcting effect of the  $\delta_i^2$  term in (6.2) was not appreciated. Should the approximation be discovered in the source code of a modern computer program, 'bad' is only one of a range of epithets that might be applied. The remaining relevance of the formula is that it can be used to recalculate the results of early investigations, provided that the raw data are not too heavily buried under the interpretation.

The bad approximation has an error of -51% when  $r = \delta_i$ , reducing to -5% when  $d/\delta_i = 20$ . For a copper wire of 0.5 mm diameter, it does not converge with the good approximation until well into the UHF range, and the error is -5% at 7 MHz. By contemporary standards, the assumption that the  $\delta_i^2$  term in the effective area can be dropped is not valid for ordinary small wires.

## 7. Early asymptotic form of the Bessel function combination

In section 5, a compact form of the exact expression for AC resistance factor was given as:

$$\Xi = \frac{rU(q)}{\delta_i\sqrt{2}}$$

Where  $U(q)$  is a combination of Bessel functions of argument  $q = \sqrt{2} r/\delta_i$ . The Bessel function calculation is highly convergent with the good thick conductor approximation (6.2) when  $q$  is large, and so we can use that information to obtain an important asymptotic form. As  $q \rightarrow \infty$ , we have from (6.2):

$$\Xi_\infty = \frac{r^2}{2r\delta_i - \delta_i^2} = \frac{r^2}{\delta_i(2r - \delta_i)}$$

Thus

$$\Xi_\infty = \frac{r}{\delta_i} \frac{r}{(2r - \delta_i)} = \frac{r}{\delta_i\sqrt{2}} \frac{r\sqrt{2}}{(2r - \delta_i)} = \frac{r}{\delta_i\sqrt{2}} U_\infty(q)$$

Hence, we identify:

$$U_\infty(q) = \frac{r\sqrt{2}}{2r - \delta_i}$$

Inverting this expression we get:

$$\frac{1}{U_\infty(q)} = \frac{2r - \delta_i}{r\sqrt{2}} = \sqrt{2} - \frac{\delta_i}{r\sqrt{2}}$$

but  $q = \sqrt{2} r/\delta_i$ , hence:

$$\frac{1}{U_\infty(q)} = \sqrt{2} - \frac{1}{q}$$

which, upon re-inversion gives:

$$U_\infty(q) = \frac{1}{\sqrt{2} - 1/q} \quad q \gg \frac{1}{\sqrt{2}} \quad \text{Early asymptotic form} \quad (7.1)$$

We will refer to this expression as the 'early' asymptotic form of  $U(q)$ , because  $U(q)$  converges with it for only moderately large values of  $q$ . We can also find a 'late' asymptotic form by noting that as  $q \rightarrow \infty$ ,  $1/q \rightarrow 0$ , and so  $U_\infty(q) \rightarrow 1/\sqrt{2}$ . The late asymptotic form is what we would have obtained had we used the traditional (bad) thick conductor approximation (6.3) as our starting point.

The significance of the early asymptotic form of  $U(q)$  (i.e., the 'good' thick-conductor approximation) is that it contains an element of correction for surface curvature, its denominator is a polynomial, and (to the author's knowledge) its importance has not been recognised elsewhere. As discussed next, it leads to a precise method for calculating  $\Xi$ , etc.; either as an end in itself or as a reference for the development of simpler formulae.

## 8. Precise calculation of AC resistance (Xirac, 0.01ppM)

When Bessel functions are used for the calculation of the AC resistance factor,  $\Xi(q)$ , either using one-line series expansions or program routines, there comes a point at which the rounding error is unacceptable. The calculation procedure should then be switched to an asymptotic form. This sets the problem of how to choose and verify an appropriate asymptotic expression and select a sensible branch point.

An extremely accurate asymptotic formula for the round wire internal impedance is given by Mingli and Yu<sup>13</sup>. Using the present notation, this takes the form:

$$\frac{Z_{int}}{\ell} = \frac{j q \rho}{2 \pi r^2 \phi(q)} \quad (8.1)$$

Where  $\phi(q)$  is a complex polynomial that can be decomposed into real and imaginary parts thus:

$$\phi(q) = \text{Por}(q) + j \text{Poi}(q)$$

where:

$$\text{Por}(q) = \frac{1}{\sqrt{2}} - \frac{0.5000008}{q} - \frac{0.0884032}{q^2} + \frac{0.000256}{q^3} + \frac{0.1417216}{q^4} + \frac{0.3833856}{q^5} + \frac{0.4194304}{q^6}$$

$$\text{Poi}(q) = \frac{1}{\sqrt{2}} - \frac{0.0000008}{q} + \frac{0.0883904}{q^2} + \frac{0.1255424}{q^3} + \frac{0.1384448}{q^4} - \frac{0.0786432}{q^5} - \frac{0.8388608}{q^6}$$

Putting (8.1) into a+jb form we get:

$$\frac{Z_{int}}{\ell} = \frac{q \rho}{2 \pi r^2} \left[ \frac{\text{Poi}(q) + j \text{Por}(q)}{\text{Por}(q)^2 + \text{Poi}(q)^2} \right] \quad (8.2)$$

from which, after making appropriate substitutions, it is easily shown that the expression in square brackets in (8.2) replaces the expression in square brackets in (3.2). Hence, the AC resistance factor (with subscript MY standing for 'Mingli-Yu') is given by:

$$\Xi_{MY} = \frac{q}{2} \left[ \frac{\text{Poi}(q)}{\text{Por}(q)^2 + \text{Poi}(q)^2} \right] \quad \pm < 0.1 \text{ppM for } q \geq 11.5 \quad (8.3)$$

where the factor in square brackets replaces the Bessel function combination  $U(q)$  of section 5. Notice also, from the polynomial expansions, that as  $q \rightarrow \infty$ ,  $\text{Por}(q) \rightarrow 1/\sqrt{2}$ ,  $\text{Poi}(q) \rightarrow 1/\sqrt{2}$ , and so  $\Xi_{MY} \rightarrow q / 2\sqrt{2}$ . This confirms that  $\Xi_{MY}$  converges ultimately with the late asymptotic form discussed in the previous section. By comparison with the optimised Bessel calculation; the error in the MY formula was found to fall within  $\pm 0.1$  ppM for  $q > 11.5$ , and within about  $\pm 0.02$  ppM at  $q = 65$ .

13 "Numerical calculations of internal impedance of solid and tubular cylindrical conductors under large parameters" W. Mingli and F. Yu (Northern Jiaotong University, School of Electrical Engineering, Beijing, China). IEE Proceedings, Generation, Transmission and Distribution. Jan. 2004, Vol 151, Issue 1, p. 67-72.



It would now be a straightforward matter to write a program routine that branches to the MY formula at some point where the rounding error in the Bessel calculation is beginning to show. That however, was not the approach chosen; the main reason being that the MY formula requires some 12 empirical coefficients of 7 decimal places each, whereas the early asymptotic form (7.1) was found to be nearly convergent with the Bessel calculation in the  $q = 80$  to  $90$  region with no empirical coefficients at all. Furthermore, although the MY formula is very accurate, there will be a small discontinuity at the branch point. Such glitches can be eliminated by adjusting polynomial coefficients, but for a high-order polynomial, any adjustment can have unforeseen consequences. Hence a relatively low-order approach that exploits the highly convergent behaviour of the early asymptotic form was adopted.

It was mentioned previously that the early asymptotic form is already the reciprocal of a polynomial. It is therefore a simple matter to extend it; i.e.:

$$U_{\infty}(q) = \frac{1}{\sqrt{2} - \frac{1}{q} + \frac{k_2}{q^2} + \frac{k_3}{q^3} + \frac{k_4}{q^4} + \dots}$$

It is also the case, that with the coefficients  $k_2$  and higher set to zero, this expression comes out a few ppM larger than the exact calculation when  $q$  is large. Consequently, noting that the polynomial is in the denominator, it is possible to nudge  $U_{\infty}$  onto the exact calculation using only positive empirical coefficients; and by such constraint, the expression is prevented from developing undulations. This means that it is possible to make the two calculation methods contiguous at some chosen branch-point, and since the empirical terms die-out rapidly with increasing  $q$ , the  $U_{\infty}$  calculation can only converge with the early asymptotic form thereafter. Thus, effectively, the calculation is made to switch over to an asymptotic form in such a way that the maximum error of the resulting program routine is the same as the rounding-error just below the branch point.

For the purpose of this investigation, comparisons were made according to the proportionate difference; this being given by the absolute difference divided by the average of the calculations being compared. Thus, for example, if we compare an asymptotic form  $\Xi_{\infty}$  with the optimised Bessel calculation  $\Xi_{OB}$ , the difference function is:

$$\Delta_{\Xi} = 2 \frac{(\Xi_{\infty} - \Xi_{OB})}{(\Xi_{\infty} + \Xi_{OB})}$$

The reason for this choice is that, over the comparison interval, the Bessel calculation is accurate but not precise, whereas the asymptotic form is precise but not accurate, so neither is authoritative. The difference in ppM is obtained by multiplying the proportionate difference by  $10^6$ .

The remaining issue is that of how to obtain near-exact contiguity at the crossover; the problem being that, although the Bessel calculation is extremely accurate, it is nevertheless noisy at the parts per  $10^9$  level. The adopted solution was to smooth the difference data in the candidate crossover region by fitting to a regression line calculated on 201 points, the  $q$ -values (abscissa) being spaced at an interval just large enough to be recognised as  $> 0$  by the program interpreter. The best estimate for the true difference was then taken as the regression value at the chosen point. Also, by varying the calculation interval, it was found that the peak magnitude of the Bessel calculation noise is not significantly affected by scale (i.e., it is fractal), which means that the worst-case error just below the branch point can be estimated from the peak error in the regression calculation.

After several experiments, a crossover point at  $q = 65$  was chosen. This decision is somewhat arbitrary; being a compromise between choosing a low crossover point, where the Bessel calculation noise will be low, but the empirical coefficients will be relatively large (and therefore

cause for doubt about convergence with the unknown exact value at higher  $q$ ); or choosing a high crossover point, where the Bessel noise will be greater but the coefficients will be small. At  $q = 65$ , the peak noise in the Bessel calculation is about 0.008 ppM (using double precision in the Open Office Basic programming environment, and avoiding the use of the exponentiation operator). This error is negligible in any practical application.

By allowing two empirical coefficients ( $k_2$  and  $k_3$ ) to vary; the Bessel calculation and the polynomial were made coincident within 0.1 ppM over the range from  $q = 29$  to  $q = 65$ , and a final small adjustment to  $k_2$  brought them into near-exact agreement at  $q = 65$ . As a check on accuracy, the polynomial was then compared with the Mingli-Yu formula for  $q > 65$ . The two were found to agree within 0.02 ppM, but since the greatest disagreement occurred just above  $q = 65$ , where the polynomial and the Bessel calculation agree, the error (trivial though it is) must be attributed to the Mingli-Yu formula (agreement within 0.01 ppM occurred for  $q > 83$ ). Hence the final overall accuracy obtained by branching for  $q > 65$  is about 8 parts in  $10^9$  as expected.

The considerations just outlined were coded into the Open Office Basic macro "Xirac", shown below; the name being coined from: " $\Xi$  (Xi) for the calculation of  $R_{ac}$ ". The various fitting and test calculations were carried out in a worksheet available from the author's website<sup>14</sup>. The program language is described in the Open Office Basic programming guide<sup>15</sup>.

Series summations for the Kelvin functions are all calculated in the same program loop, using the recursion relations described by Bob Weaver<sup>16</sup>. These are:

$$\text{For Ber}(q): \quad \text{term}(0) = 1 \quad , \quad \text{term}(n) = -\text{term}(n-1) \times [q/(4n)]^2 \times [q/(4n-2)]^2$$

$$\text{For Bei}(q): \quad \text{term}(0) = q^2/4 \quad , \quad \text{term}(n) = -\text{term}(n-1) \times [q/(4n+2)]^2 \times [q/(4n)]^2$$

$$\text{For Ber}'(q): \quad \text{term}(0) = -q^3/16 \quad , \quad \text{term}(n) = -\text{term}(n-1) \times [q/(4n+4)] \times [q/(4n+2)]^2 \times [q/(4n)]$$

$$\text{For Bei}'(q): \quad \text{term}(0) = q/2 \quad , \quad \text{term}(n) = -\text{term}(n-1) \times [q/(4n+2)] \times [q/(4n)]^2 \times [q/(4n-2)]$$

Where  $\text{term}(n)$  is the  $n^{\text{th}}$  term in the series, and  $n$  begins at 0. Notice, that for a given series length, the four functions are of different order, with the maximum power increasing in the sequence: Ber, Bei', Bei, Ber'. High power terms vanish before low power terms; and so Ber should be expected to require the greatest number of iterations, and Ber' the least. This proposition was checked by writing a function to return the number of iterations required for a given convergence criterion. It was found, for  $q$  up to 100, that the four functions often require the same number of iterations; Bei' never requires more terms than Ber; and sometimes Ber and Bei' require one or two more terms than the others. Hence the convergence of the Ber series is a suitable test for overall convergence. There is little computational overhead in sometimes calculating more terms than necessary, and some saving in the pre-calculation of recursion factors common to all four functions. With the convergence criterion shown in the program, the maximum number of iterations required for  $q$  up to 65 is 39 (i.e., 40 terms in the series).

14 <http://g3ynh.info/zdocs/comps/zint/index.html>

See spreadsheet **Xirac.ods** (sheets 1-3). The function can be accessed (for copying to other spreadsheets, etc.) via the 'Tools > Macros > Organise Macros > OOo Basic' menu on the Open Office header.

15 [http://wiki.services.openoffice.org/wiki/Documentation/BASIC\\_Guide](http://wiki.services.openoffice.org/wiki/Documentation/BASIC_Guide)

16 **Practical considerations in the calculation of Kelvin Bessel functions and complete elliptic integrals  $K$  and  $E$** . Robert S Weaver, 2009. (Citation given earlier).

```

Function Xirac(ByVal q as double) as double
'Calculates AC resistance factor Xi within 0.01 ppm. Version 2.00.
  Dim n%, a#, b#, c#, br#, bi#, dr#, di#, brtm#, bitm#, drtm#, ditm#
  if q=0 then
    Xirac=1
  elseif q>65 then
    'Use early-asymptotic polynomial.
    Xirac=(q/2)/(sqr(2)-1/q+0.176417/q^2+0.280347/q^3)
  else
    'Perform Kelvin Bessel func. calc. Start by setting up initial term values.
    brtm=1
    bitm=q*q/4
    drtm=-q*q*q/16
    ditm=q/2
    'Load initial values into summation variables.
    br=1
    bi=bitm
    dr=drtm
    di=ditm
    n=0
    do
      n=n+1
    'Calculate common recursion factors.
    a=q/(4*n-2)
    b=q/(4*n)
    c=q/(4*n+2)
    'Calculate new terms using recursion relations.
    brtm=-brtm*a*a*b*b
    bitm=-bitm*b*b*c*c
    drtm=-drtm*b*c*c*q/(4*n+4)
    ditm=-ditm*a*b*b*c
    'Add new terms to the summations.
    br=br+brtm
    bi=bi+bitm
    dr=dr+drtm
    di=di+ditm
    'Convergence test. Ber has lowest order for a given no. of terms, so needs most iterations.
    loop until abs(brtm/br)<1E-12
    Xirac=(q/2)*(br*di-bi*dr)/(dr*dr+di*di)
  end if
end function

```

Note that there is also an older version of Xirac<sup>17</sup>. That version branches to the early asymptotic form (without empirical coefficients) at the point where the noise in the Bessel calculation is equal to the difference between the two calculation methods (the crossover point for minimum discontinuity, which is at  $q \approx 89.56$ ). The maximum error in that case is 16 ppm. While continued use of the old function is not harmful, its replacement by the new one given above is recommended.

<sup>17</sup> Given in version 1.00 of this report (Dec. 2009). See sheets 4 and 5 of the worksheet **Xirac.ods**

## 9. Truncated exponential decay

The approach developed in the previous section makes it possible to calculate AC resistance with high precision using freely available software. That method however, will not necessarily translate easily into other circuit-modelling environments and can be somewhat slow in computation. Hence there is value in calculation methods that provide sufficient rather than extreme accuracy. With that in mind, note that the only common metals worth using as conductors for radio-frequency applications are Copper, Silver and Aluminium. Those materials have temperature coefficients of volume resistivity in the region of 3800 to 3930 ppM/°C at 20 °C<sup>18</sup> (i.e., about 0.4% /°C). Hence, given that temperature is not usually considered (and resistivity is thus only nominal), many investigators will be satisfied with formulae good to a few %. A major caveat however lies with the habit of failing to question the approximation used and thereby straying outside the applicable range.

For RF circuit modelling purposes, the early asymptotic (good) thick-conductor approximation discussed in section 6 is a considerable improvement over the traditional formula. The danger in using it however, is that it explodes in the region of  $d = \delta_i$ . Hence the first step in improving it is to find a modification that gives the correct lower limit:

$$R_{ac} \rightarrow R_{dc} \text{ as } d/\delta_i \rightarrow 0$$

This can be done by multiplying the thick-conductor skin depth  $\delta_i$  by a function chosen so that the product has a value  $r$  when  $d/\delta_i \rightarrow 0$ . If we call this modified skin depth  $\delta_i'$ , then equation (6.1) becomes:

$$A_{eff} = \pi(2r\delta_i' - \delta_i'^2) = \pi(d\delta_i' - \delta_i'^2)$$

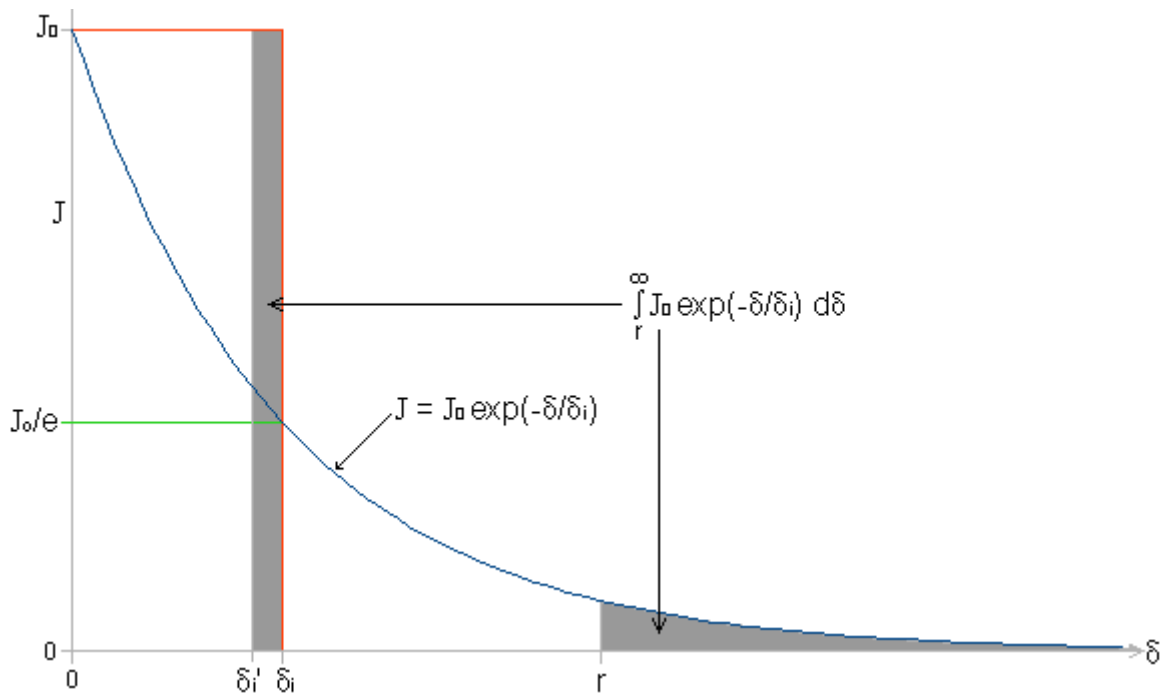
and if  $\delta_i' \rightarrow r$  when  $d/\delta_i$  is small, then:

$$A_{eff} \rightarrow \pi r^2$$

A solution can be found by considering the derivation of skin depth, as discussed in section 6.  $\delta_i$  is related to the area of a rectangle having a dimension in common with, and the same area as, an exponential decay curve. Hence, to a first approximation, the modified skin depth can be related to the area under a decay curve truncated at finite depth. The idea is illustrated below, where the area lost by truncation is subtracted from the area that determines  $\delta_i$ , resulting in a new skin depth  $\delta_i'$ .

---

18 **CRC Handbook of Chemistry and Physics**, 63rd edition. 1982-83 (CRC press, Florida) [Newer editions exist].  
E81: Resistivity of metals.



It is not obvious that the truncation should occur at depth  $r$  (the wire axis), but proof will be given shortly. For now, observe that the area  $J_0 \delta_i'$  must be the same as the area under the decay curve from  $0$  to  $r$ . Thus:

$$J_0 \delta_i' = \int_0^r J_0 \exp\left(-\frac{\delta}{\delta_i}\right) d\delta = J_0 (-\delta_i) \left[ \exp\left(-\frac{r}{\delta_i}\right) - 1 \right]$$

and the modified skin depth is:

$$\delta_i' = \delta_i \left[ 1 - \exp\left(-\frac{r}{\delta_i}\right) \right] \quad (9.1)$$

As a first check on this solution, note that as  $r \rightarrow \infty$ ,  $\exp(-r/\delta_i) \rightarrow 0$  and so  $\delta_i' \rightarrow \delta_i$ . The equation converges with the thick-conductor approximation as required. The behaviour as  $f \rightarrow 0$  and  $\delta_i \rightarrow \infty$  however, is a little more difficult to determine. If we take the equation at face value, it says that as  $\delta_i \rightarrow \infty$ ,  $\delta_i' \rightarrow 0 \times \infty$ ; i.e.,  $\delta_i'$  is indeterminate. This is reasonable however; because  $\delta_i'$  is a function of frequency, and zero frequency can be approached but never reached. We can only discover the true limiting condition by moving away from  $f = 0$  by an infinitesimal amount, and that can be done by making use of the series expansion for  $e^x$ :

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots$$

When  $|x|$  is small, the high-order terms vanish, leaving:

$$e^x \approx 1 + x$$

Hence, for the present problem, as  $\delta_i \rightarrow \infty$ ,  $\delta_i' \rightarrow \delta_i \left(1 - 1 + \frac{r}{\delta_i}\right)$ ,

i.e.,

$$\delta_i' \rightarrow r$$

The modified skin depth is the same as the wire radius at low frequencies. Hence equation (9.1) satisfies the necessary limiting conditions, and an asymptotically correct formula for the effective area is:

$$A_{\text{eff}} = \pi (2 r \delta_i' - \delta_i'^2) = \pi (d \delta_i' - \delta_i'^2)$$

where  $\delta_i' = \delta_i \left[1 - \exp\left(-\frac{r}{\delta_i}\right)\right]$  TED (9.2)

For obvious reasons, this is referred to here as the 'truncated exponential decay' (TED) approximation.

It is worth noting that the function

$$x' = x \left[1 - \exp\left(-\frac{a}{x}\right)\right]$$

can be used to correct any formula that behaves incorrectly for  $x \rightarrow \infty$ . We can, for example, also use it to correct the traditional (bad) thick conductor approximation:

$$A_{\text{eff}} = 2\pi r \delta_i$$

In this case, we want  $A_{\text{eff}} \rightarrow \pi r^2$  when  $\delta_i \rightarrow \infty$ , and since

$$\left[1 - \exp\left(-\frac{a}{\delta_i}\right)\right] \rightarrow \frac{a}{\delta_i} \quad \text{when } \delta_i \rightarrow \infty$$

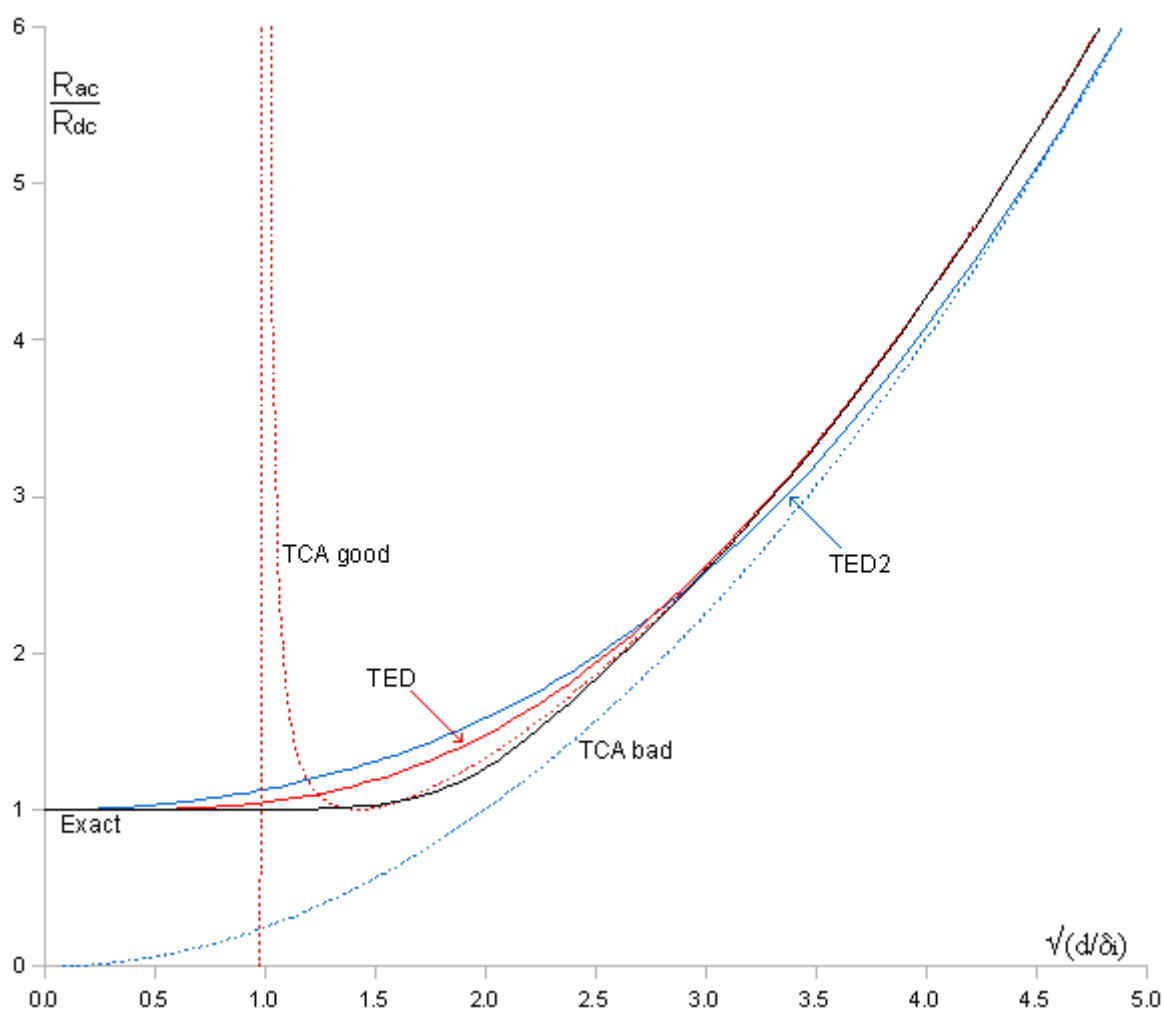
we obtain the required limiting behaviour when  $a = r/2$ . Hence, an asymptotically correct formula based on the bad thick-conductor approximation is:

$$A_{\text{eff}} = 2\pi r \delta_i'' = \pi d \delta_i'' \quad \text{TED2 (bad) (9.3)}$$

$$\text{where } \delta_i'' = \delta_i \left[1 - \exp\left(-\frac{r}{2\delta_i}\right)\right]$$

The two TED approximations are compared with the exact calculation in the graphs below. Also shown dotted are the thick-conductor approximations (TCAs) from which the doubly asymptotic forms were obtained<sup>19</sup>.

<sup>19</sup> See worksheet: **Xi\_aca.ods**, sheet 1, available from <http://g3ynh.info/zdocs/comps/zint/index.html>



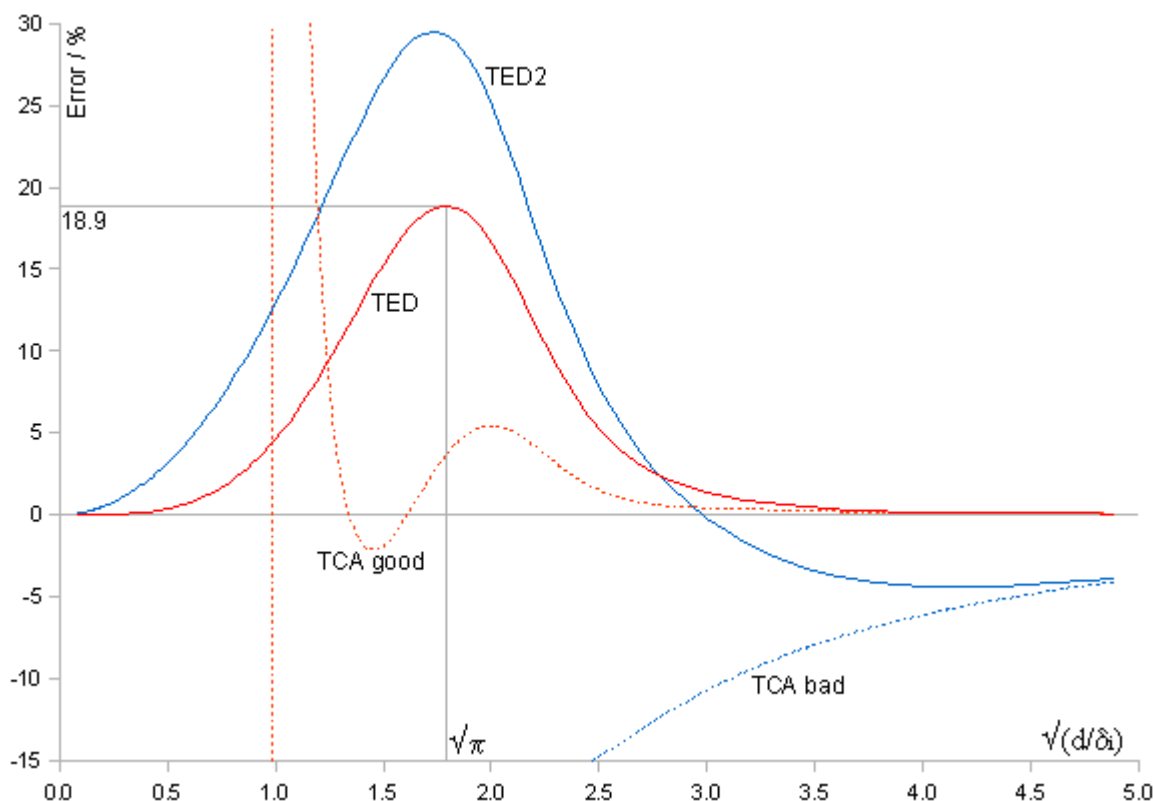
The goodness of an approximation is demonstrated by plotting the difference between it and the exact value, as in the graph below. The curves shown are all calculated from the expression:

$$\text{Error} / \% = 100 \left[ \frac{\overline{E}_{\text{approx}}}{\overline{E}_{\text{exact}}} - 1 \right]$$

The TED2 approximation (9.3), as might have been expected, shows little promise for further improvement. It has the correct limiting behaviour, but only in the sense that it converges with the bad TCA at high frequencies and will eventually converge with the exact calculation at UHF. Except for the crossover point, it fails to approach the zero error line in the range of interest for practical impedance calculations; and it will therefore be extremely difficult to correct.

The TED approximation obtained from the good TCA however, is remarkably well-behaved. Plotted on a scale proportional to the fourth-root of frequency (as above); the difference curve has the form of a nearly-symmetric lineshape function or a normal distribution, with principal parameters intriguingly related to mathematical constants; i.e., the peak occurs when:

$$\frac{d}{\delta_i} = \pi \quad \text{and} \quad \frac{\overline{E}_{\text{ted}}}{\overline{E}_{\text{exact}}} = 2^{1/4} = 1.189 \quad .$$



## 10. Variable-order limit correction

The low-frequency limiting condition can also be satisfied by a modified version of equation (9.1), specifically:

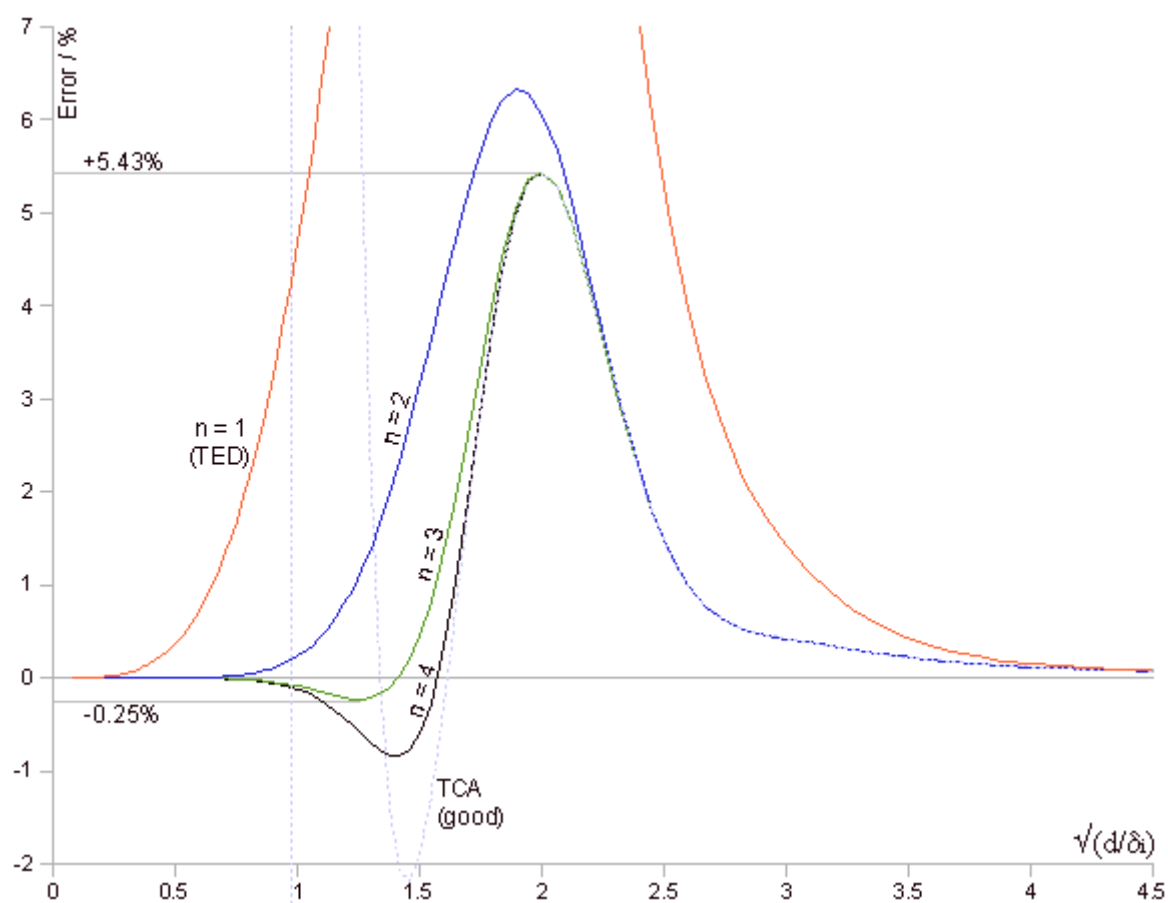
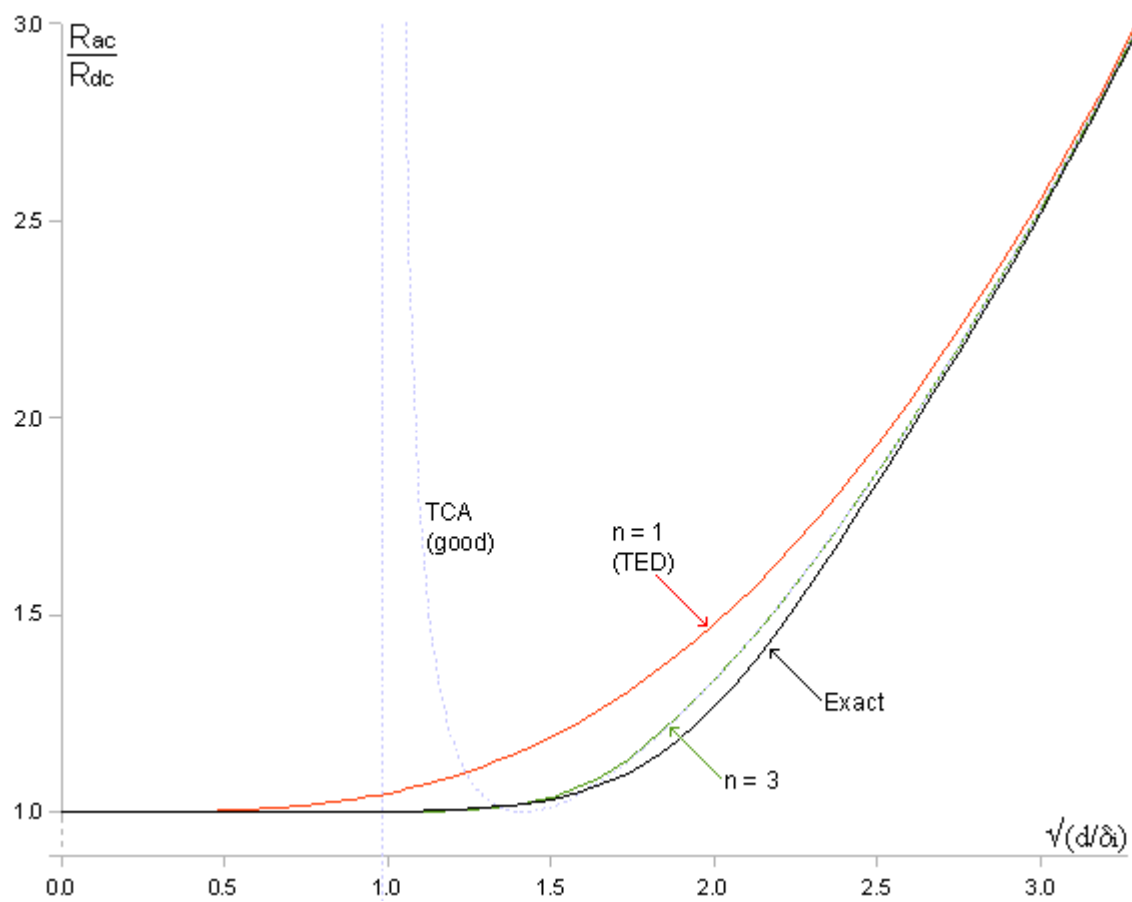
$$\delta_{ni} = \delta_i \left[ 1 - \exp \left( - \left[ \frac{r}{\delta_i} \right]^n \right) \right]^{\frac{1}{n}} \quad (10.1)$$

Now the quantity inside the square bracket goes to  $(r/\delta_i)^n$  as  $\delta_i \rightarrow \infty$ , but taking the  $n^{\text{th}}$  root restores it to  $r/\delta_i$ , so that  $\delta_{ni} \rightarrow r$ . This type of correction will be referred to as ACA $n$  (asymptotically-correct approximation of order  $n$ ). Note that (10.1) is the same as (9.1) when  $n = 1$ , and so ACA1 is the same as TED.

Shown below is a comparison between the exact calculation, TED, ACA3, and the good TCA on which the other approximations are based<sup>20</sup>. The effect of raising the order ( $n$ ) is to reduce the influence of the correcting function for higher values of  $d/\delta_i$ . This causes  $\delta_{ni}$  to converge with  $\delta_i$  much earlier than it otherwise would have done. The quantitative outcome is shown in the set of error curves (lower graph).

<sup>20</sup> See worksheet **Xi\_aca.ods**, sheet 2





The graphs show that if the ACA is to be used without further correction, the best integer choice for  $n$  is 3. Increasing the order beyond 3 has no effect on the maximum positive error, but starts to increase the maximum negative error. Hence, a continuous formula for  $A_{\text{eff}}$  accurate to within 5.5% is:

### Rac-ACA3 (5.5%)

|   |                  |      |               |
|---|------------------|------|---------------|
| $A_{\text{eff}} = \pi(2r\delta_{3i} - \delta_{3i}^2) = \pi(d\delta_{3i} - \delta_{3i}^2)$                           | +5.43%<br>-0.25% |      |               |
| where $\delta_{3i} = \delta_i \left[ 1 - \exp\left(-\left[\frac{r}{\delta_i}\right]^3\right) \right]^{\frac{1}{3}}$ |                  | ACA3 | <b>(10.2)</b> |

## 11. Secondary correction methods

For a formula,  $\Xi$  say, for which a first approximation  $\Xi_1$  exists, and for which the exact solution is also known, we can define an error function:

$$y_{(\text{obs})} = \frac{\Xi_1}{\Xi} - 1 \quad (11.1)$$

(where 'obs' stands for 'observed'). Other choices are possible, but this has the virtue that it vanishes when there is no error. Hence an exact solution for  $\Xi$  can be written:

$$\Xi = \frac{\Xi_1}{1 + y_{(\text{obs})}}$$

In some trivial cases, an analytical solution for  $y_{(\text{obs})}$  will exist; but in general, an improved second approximation can be obtained by finding a function  $y$  that is a good approximation to  $y_{(\text{obs})}$ . The new formula is then:

$$\Xi_2 = \frac{\Xi_1}{1 + y} \quad \text{Divisor correction}$$

Notice that when the error in  $\Xi_1$  is positive, then  $y$  is positive, and the factor  $1/(1+y)$  is less than 1 (and vice versa). Hence, the effect of the correction is to multiply  $\Xi_1$  by a factor that adjusts it in an attempt to reproduce the correct value  $\Xi$ . It follows that we can define an alternative correction function  $y'$  such that:

$$\Xi_2 = \Xi_1 (1 - y') \quad \text{Multiplier correction}$$

$y'$  will be similar to  $y$ , but not identical. There will also be a corresponding  $y'_{(\text{obs})}$ , obtained from the expression:

$$\Xi = \Xi_1 (1 - y'_{(\text{obs})})$$

Hence:

$$y'_{(\text{obs})} = 1 - \frac{\epsilon}{\epsilon_1} \quad (11.2)$$

which also vanishes when there is no error.

It is possible to find an expression for  $y$  or  $y'$  by least-squares fitting to some kind of polynomial. Were that the intention however, then there would be little merit in working from an initial approximation; because the investigation might just as well proceed by fitting the original function  $\Xi$  to a polynomial. The point in applying successive corrections is to avoid brute-force methods; which are at best inelegant, and often result in expressions that cannot be extrapolated and are difficult to transcribe. A least-squares fit moreover, is not what is required; since the intention is to obtain the smallest possible maximum error, rather than to minimise the RMS error.

An important consideration also, is to find a correction function of relatively low order; i.e., a function that does not contain terms of the argument raised to a high power and therefore cannot change direction suddenly. The issue here is that the approximation can only be compared with the exact function at a finite number of points during development, but the final formula must return a value within stated limits for any argument supplied.

Hence the fitting method used is essentially graphical; a laborious business in the days before electronic computers, but a task for which spreadsheet programs (for example) are eminently suitable. The approach is to obtain various initial approximations, as has been done in the preceding sections; and select one, which is not necessarily the most accurate, but which has an error curve resembling a simple mathematical function. That function then becomes the correcting function ( $y$  or  $y'$ ); and as discussed above, there are two ways in which it can be applied, and the final choice is dictated by whichever gives the best result. Initial parameter adjustment is by hand, but once the curve match begins to look good, a non-linear fitting routine can be used for final optimisation.

One additional operational point worth emphasising, is that the appearance of an error function (and therefore its relationship to the candidate fitting function) changes according to the scale on which it is plotted. Linear and logarithmic scales are the obvious starting point, but squares, square roots, and other powers of the argument are the next line of attack. Once a promising shape is obtained, the operation performed on the argument in order to obtain an  $x$  value is also applied to the argument inserted into the correction function. In that way, the correction takes place in a space suited to the correction process, but is automatically applied in the space in which the final formula is to be used. In other words, if the exact formula and the first approximation have an argument  $q$  (say), but the error function has to be plotted on a scale of  $x=f(q)$  to achieve a promising form, then the argument of the correcting function is  $f(q)$ , not  $q$ . Such is the general situation, but as mentioned in **section 2**, the modified Lorentzian (ML) correcting function can absorb transformations involving multiplication and raising to a power into its parameters; and it so happens that those are exactly the transformations required for the present purpose.

## 12. Modified Lorentzian correction for TED

The error function for the TED approximation, as discussed in section 9, is reminiscent of a normal distribution or a lineshape function when plotted on a scale of  $x = \sqrt{d/\delta_1}$ . Both the Gaussian and the Lorentzian however are symmetric about the mean, and can take on finite values for  $x < 0$ ; whereas the error function for an asymptotically correct approximation must be zero at  $x=0$  (and also when  $x \rightarrow \infty$ ), and the TED error function in particular is slightly skewed to the 'high frequency' side (in the manner of an electrical resonance curve). Hence modified Gaussian and Lorentzian functions, having the correct zero limit and variable skew, are the best candidates (as was discussed in section 2). A modified Gaussian function brought the maximum error to within  $\pm 0.5\%$  after machine optimisation; but will not be discussed here because it was found that all of the error curves encountered in this work are better described by a Modified Lorentzian (ML) function of the general form:

$$y = \frac{h}{\left[ 1 + w \left[ \left( \frac{x}{\bar{x}} \right)^{p1} - \left( \frac{x}{\bar{x}} \right)^{p2} \right]^2 \right]^p}$$

Note that from the graphical investigation of section 9, we already know that, at the peak of the error curve:

$$\frac{\Xi_{ted}}{\Xi} = 2^{1/4}$$

Hence if we apply a divisor correction:

$$\Xi_{ml} = \frac{\Xi_{ted}}{1 + y}$$

then we need to multiply  $\Xi_{ted}$  by  $1/2^{1/4}$  at the peak in order to make  $\Xi_{ml}$  equal to  $\Xi$ . Now recall that the maximum height of the ML function is  $h$ . Hence:

$$\frac{1}{1 + h} = \frac{1}{2^{0.25}}$$

and so, for divisor correction:

$$h = 2^{0.25} - 1 = 0.1892$$

Conversely, if we apply a multiplier correction:

$$\Xi'_{ml} = \Xi_{ted} (1 - y')$$

then

$$1 - h' = \frac{1}{2^{1/4}}$$

and so:

$$h' = 1 - \frac{1}{2^{1/4}} = 0.1591$$

The condition  $y = h$  occurs when  $x/\bar{x} = 1$ . The peak in the error curve is found to occur when

$$\sqrt{\frac{d}{\delta_i}} = \sqrt{\pi}$$

Hence

$$\frac{x}{\bar{x}} = \sqrt{\frac{d}{\delta_i \pi}}$$

The basic modified Lorentzian, minimally skewed to establish the condition:  $y=0$  when  $x=0$  (i.e., the generic form of the electrical resonance curve) is:

$$y = \frac{h}{1 + w \left[ \left( \frac{x}{\bar{x}} \right)^1 - \left( \frac{x}{\bar{x}} \right)^{-1} \right]^2}$$

Hence, if we define

$$z = \frac{d}{\delta_i \pi}$$

then the first candidate for the correction function is:

$$y = \frac{h}{\left[ 1 + w \left[ z^{1/2} - z^{-1/2} \right]^2 \right]^1}$$

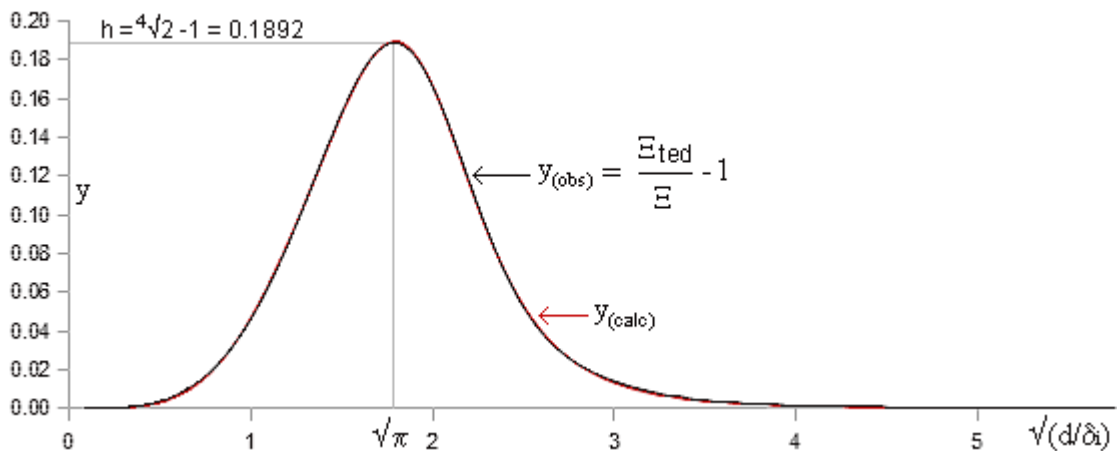
where  $h$  is already known (and depends on how the correction is applied), and  $w$  can easily be adjusted by hand to obtain a fairly good fit (starting with  $w = 1$ ).

Although the problem has at least four parameters in need of determination ( $w$ ,  $p$ ,  $p_1$ ,  $p_2$ ), an acceptable degree of optimisation was obtained by hand using a suitably organised spreadsheet. The technique used involved monitoring two graphs: one being a plot of the correction function superimposed on the error function, the other being a plot of the percentage difference between the new approximation and the exact solution. For a given correcting function, adjustable parameters were placed at the head of the column, rather than embedded in the formula, so that the effect of changing a parameter could be seen by changing a number in a single cell (this arrangement is also required for machine optimisation, see below). Movable error-limit bars were also placed on the percentage difference graph, their purpose being to indicate whether an adjustment had made matters better or worse.

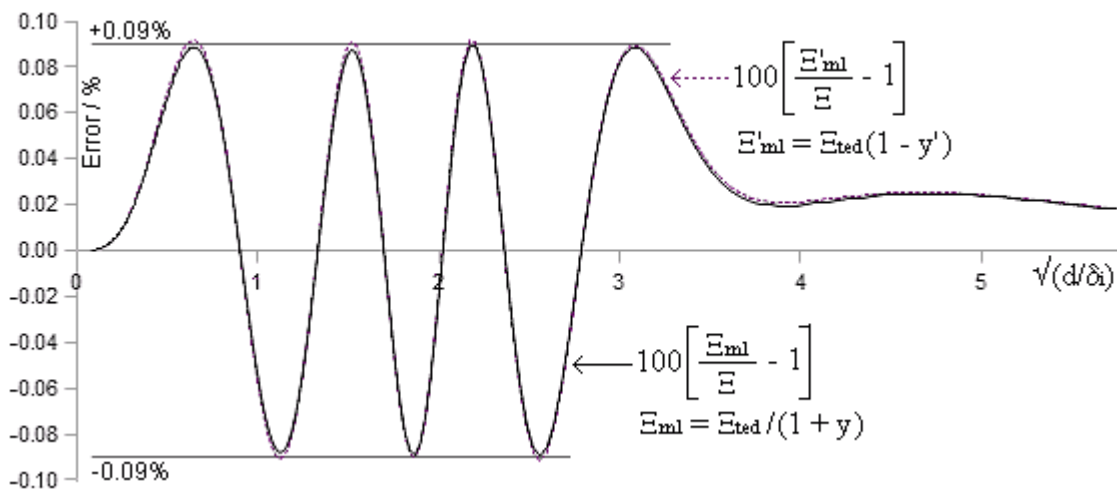
After initial adjustment, the fitting parameters were passed over to a non-linear optimisation program. For the work discussed in this article, the macro program; 'Multi-Variable Optimiser' (MVO, also referred to as 'Solver') written by Bob Weaver was used. Solver uses the Nelder-Mead downhill simplex algorithm to find minima in a multi-dimensional surface, and for this it needs a composite error function to tell it the height of the terrain. In the case where an approximation

formula is optimised against an exact calculation method, the correct fitting criterion is that of minimum runout; i.e., we want to guarantee that the resulting formula will always produce a result that lies between strictly defined error limits. A proportionate error criterion was chosen, so that the program would minimise percentage error, and progressive weighting was used to make the error boundaries parallel. These issues are discussed in another article<sup>21</sup>.

It turned out, for this case, that the best fit was obtained using divisor correction. The correcting function is shown below, superimposed on the error curve obtained by comparing the exact formula against TED<sup>22</sup>. Note that although  $h$  and  $\bar{x}$  are defined using mathematical constants, they were allowed to vary in the fit, giving a small improvement in the overall maximum runout by allowing them to deviate slightly from their 'natural' values. After fitting, the parameters were rounded to 6 or less decimal places and adjusted in the last decimal place to re-minimise the error.



The goodness of the correction is shown in the overall error graph below. Also shown dotted is the error in the multiplier-corrected formula; which is only very slightly inferior. The author is unaware of any straightforward method that will decide in advance which will be best, and so both approaches had to be tried. Note the vertical axis. The curves do not represent chaotic behaviour, but tiny deviations from the exact result. The final error bars are placed at  $\pm 0.09\%$ , this being the guaranteed worst-case error in using the better of the two formulae.



21 **Optimisation of multi-parameter empirical fitting functions.** D W Knight [www.g3ynh.info - Data analysis]

22 The details are in worksheet **Xi\_ted-ml.ods**,

The complete TED-ML recipe for calculating  $\Xi$  is shown below.

### Rac-TED-ML (0.09%)

$$\Xi = \frac{R_{ac}}{R_{dc}} = \frac{r^2}{(2r\delta_i' - \delta_i'^2)(1+y)} \quad \pm 0.09\%$$

$$\delta_i' = \delta_i \left[ 1 - \exp\left(-\frac{r}{\delta_i}\right) \right] ,$$

$$\delta_i = \sqrt{\frac{\rho}{\pi f \mu}} ,$$

$$y = \frac{0.189774}{(1 + 0.272481 [ z^{1.82938} - z^{-0.99457} ]^2)^{1.0941}} ,$$

$$z = 0.62006 \frac{r}{\delta_i} .$$

The formula is also shown below coded as an Open Office Basic macro function, using the input argument q (as for the Bessel calculation). This function was written for the purpose of comparing various calculation methods (see section 15); but internally it uses the argument:

$$q' = \frac{q}{\sqrt{2}} = \frac{r}{\delta_i}$$

```
Function Xitml(byval q as double) as double
'AC resistance factor. TED-MLD formula (+/-0.09%).
Dim z as double, s as double, qprime as double, y as double
if q=0 then
  xitml = 1
else
  qprime = q / sqr(2)
  s = (1-exp(-qprime))/qprime
  xitml = 1/(2*s - s*s)
  z = 0.62006*qprime
  y = 0.189774 / (1+0.272481*(z^1.82938-z^-0.99457)^2)^1.0941
  xitml = xitml/(1+y)
end if
end function
```

### 13. ML correction for variable-order ACA

In section 10, an improvement was obtained over the initial TED approximation by using an expression for modified skin-depth of the form:

$$\delta_{ni} = \delta_i \left[ 1 - \exp \left( - \left[ \frac{r}{\delta_i} \right]^n \right) \right]^{\frac{1}{n}}$$

where  $n$  was referred to as the 'order'. The additional parameter  $n$  can also be included and allowed to vary when carrying out ML correction; subject to the condition that the difference between the first approximation and the exact curve is roughly Lorentzian and does not cross zero to any significant extent.

In the hand-optimisation phase of this work, it was assumed that an initial minimisation by adjusting  $n$  prior to ML correction would produce the best outcome. That view seemed to be corroborated when a hand-adjustment of TED-ML ( $n=1$ ) produced a maximum runout of 0.28%, whereas a hand adjustment with  $n=2.5$  (about the highest order that can be used without causing the difference function to cross the  $y$  axis) reduced the runout to 0.16%. Subsequent machine optimisation produced a different outcome however. When starting with  $n=2.5$ , the parameter was found to drift up to about 2.56, and the runout reduced to just under 0.15%. Starting with  $n=1$ , the parameter remained at about 1, and the runout reduced to just under 0.09% as for the TED case discussed in the previous section. It was found that the extra parameter improved the fit by only about 3 parts in  $10^5$ . This is not sufficient to warrant its inclusion<sup>23</sup>.

It transpires, in the AC resistance case, that the correction process 'prefers' the TED approximation over the variable-order ACA. We can also observe that the arbitrary skirt-modifying parameter  $p$  in the ML function given in the previous section is about 1, which implies that the difference function for TED is an actual Lorentzian (in second order) after suitable transformation of the input argument.

---

<sup>23</sup> The ACA-ML calculations are given in the worksheet: **Xi\_aca-ml.ods** (sheets 1-3). They do not lead to any significant improvement over the TED-ML formula of the previous section.



## 14. Effective relative skin depth

In section 12, a formula for AC resistance was obtained by applying the TED correction to the skin depth inserted into the good thick-conductor approximation, and then applying ML correction to the resulting approximation for the AC resistance factor. There is an alternative approach; which is that of applying all of the correction to the skin depth, so that the good TCA then yields an accurate value for AC resistance directly. Note incidentally, that the good TCA (section 6) is that which gives the proper relationship between effective conductor cross-sectional area and skin depth, and so the effective skin depth that must be fed into it is that which is most rigorously defined insofar as the abrupt conduction cutoff model is valid. The exact 'effective' skin depth ( $\delta_{i(\text{eff})}$ ) say) is thus defined in the following relationship:

$$\Xi = \frac{R_{ac}}{R_{dc}} = \frac{r^2}{2r\delta_{i(\text{eff})} - \delta_{i(\text{eff})}^2}$$

To solve this expression for  $\delta_{i(\text{eff})}$ , first take the reciprocal:

$$\frac{1}{\Xi} = \frac{2\delta_{i(\text{eff})}}{r} - \frac{\delta_{i(\text{eff})}^2}{r^2}$$

It will now be convenient to define a new quantity; the effective relative skin depth,  $s$ , as:

$$s = \frac{\delta_{i(\text{eff})}}{r}$$

Note that when  $f \rightarrow 0$ ,  $\delta_{i(\text{eff})} \rightarrow r$ , and when  $f \rightarrow \infty$ ,  $\delta_{i(\text{eff})} \rightarrow 0$ . Hence  $s$  varies between 1 at low frequencies, and zero at infinite frequency. Now we have:

$$\frac{1}{\Xi} = 2s - s^2$$

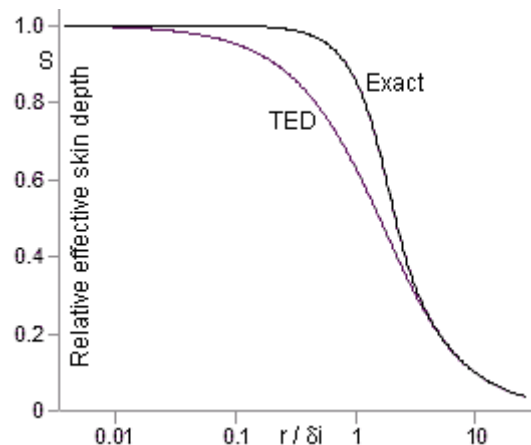
or

$$s^2 - 2s + \frac{1}{\Xi} = 0$$

which is a quadratic equation with  $a=1$ ,  $b=-2$  and  $c=1/\Xi$ . Of the two possible solutions, it is the one involving the negative of the square root that gives  $s$  varying between 1 and 0; i.e.:

$$s = 1 - \sqrt{1 - \frac{1}{\Xi}} \quad (14.1)$$

This expression can be used to calculate  $s$  from the 'exact' formula for  $\Xi$  (i.e. using function Xirac). It is shown on the right for comparison with the curve produced by the TED approximation, which is defined as:



$$s_{\text{ted}} = \frac{\delta_i'}{r} = \frac{\delta_i \left[ 1 - \exp\left(-\frac{r}{\delta_i}\right) \right]}{r}$$

Notice that the difference between the exact and the TED functions will be a bell-shaped curve. Hence the TED approximation can be subjected to ML correction to produce a more accurate approximation for  $s$ . As before, both divisor and multiplier corrections were tried, and the divisor method gave the best result by a tiny margin<sup>24</sup>. Thus we have:

$$s_{\text{ted-ml}} = \frac{s_{\text{ted}}}{1+y}$$

The determined parameters for the ML function  $y$  are as given in the box below, which shows a Basic macro function for calculating  $s$ . As before, the internal argument is  $q' = r/\delta_i$ .

```
Function Sktml(byval q as double) as double
'Relative effective skin depth. TED-MLD formula.
Dim z as double, s1 as double, qprime as double, y as double
if q=0 then
  Sktml = 1
else
  qprime = q / sqr(2)
  s1 = (1-exp(-qprime))/qprime
  z = 0.796553*qprime
  y = -0.270301 / (1+0.072481*(z^2.221576-z^-1.04095)^2)^0.945171
  Sktml = s1/(1+y)
end if
end function
```

Using the  $s$  value returned by the function above, the AC resistance factor is given as:

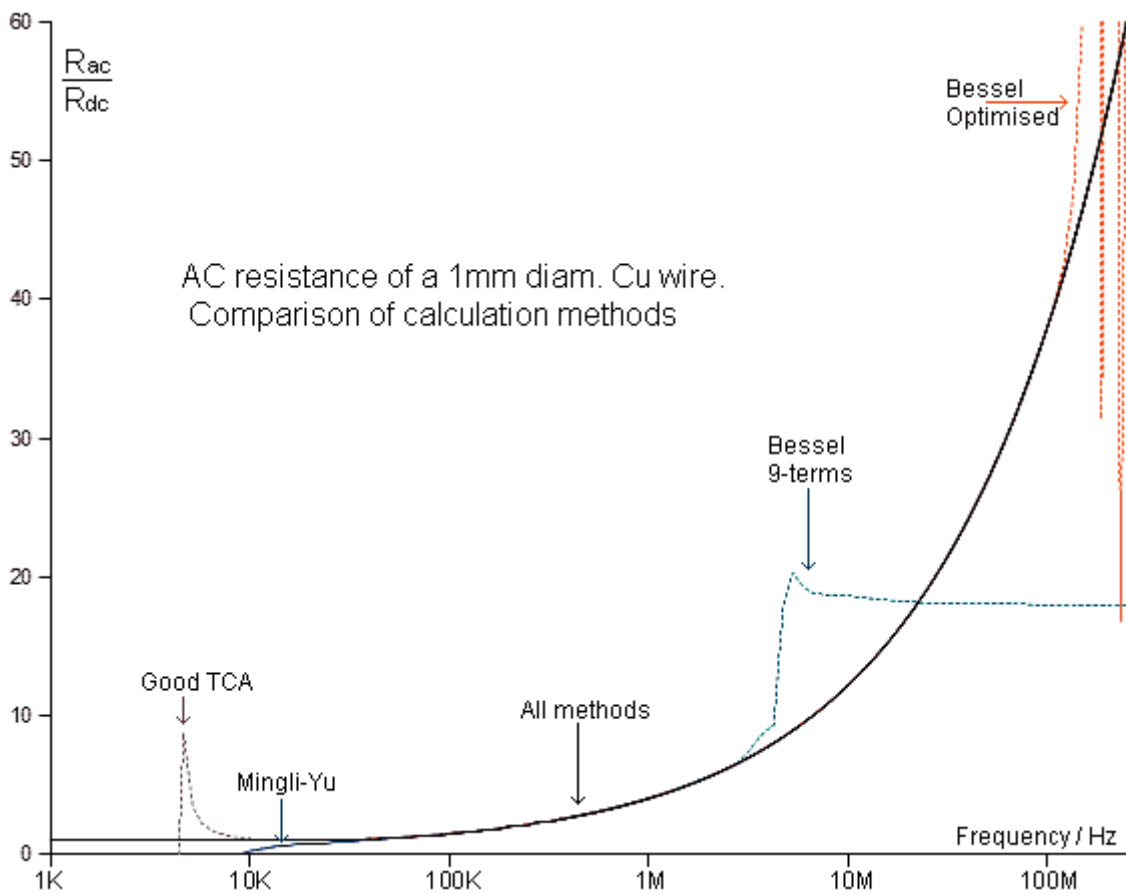
$$\Xi = \frac{1}{2s - s^2}$$

The accuracy is  $\pm 0.091\%$ .

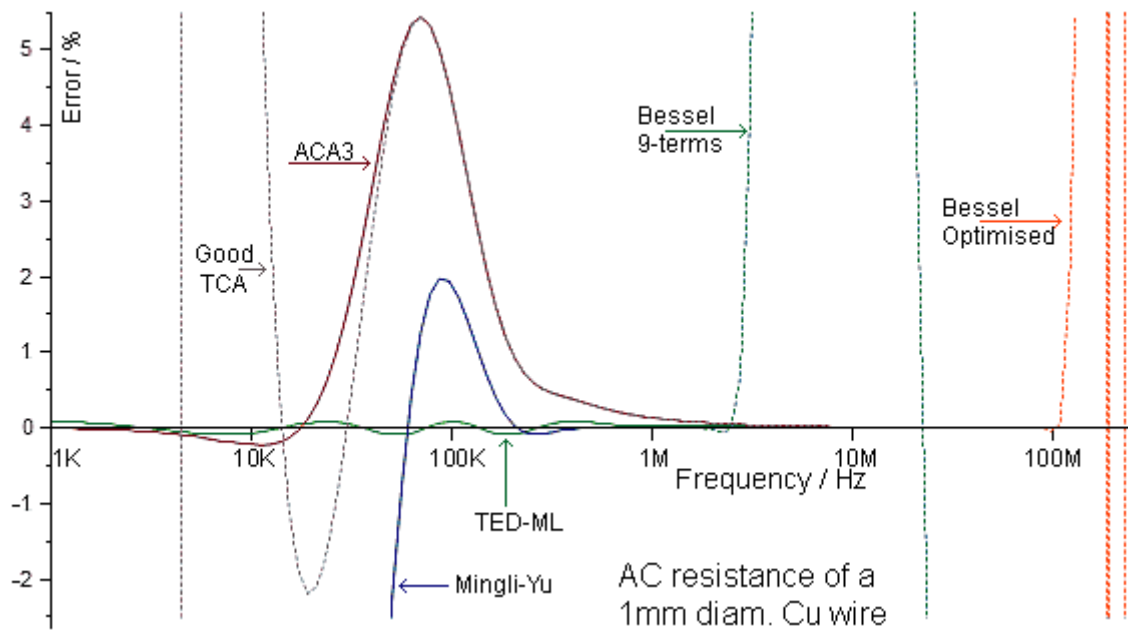
<sup>24</sup> The calculations are given in the spreadsheet: **eff\_skd.ods** (sheets 1-3).

## 15. Comparison of calculation methods

Shown below is a comparison of the AC resistance calculation methods discussed in the foregoing sections<sup>25</sup>. The scale of the first graph is such that all methods appear superimposed except for the breakdown of the good (early asymptotic) thick-conductor approximation and the Mingli-Yu formula at low frequencies, and the Bessel function calculations at high frequencies. The graph, as shown, is for a 1mm diameter copper wire. Decreasing the wire diameter moves the pattern to higher frequency (and vice versa), i.e., it changes the frequency scale, but not the appearance of the graph. The second graph shows the percentage error in a given method. The zero error reference is the function 'Xirac' discussed in section 8, which is accurate to within 1 part in  $10^8$  ( $\pm < 0.01$  ppm).



<sup>25</sup> see worksheet: **Zint\_calcs.ods**, sheet 1



## Internal Inductance

### 16. Exact expression for internal inductance

The internal inductance of a conductor is related to the energy stored in the internal magnetic field. Since the effective cross-sectional area available for conduction diminishes as the frequency increases, then so too does the internal magnetic energy. Hence internal reactance is related to the skin effect; the decrease in inductance combined with the increase in losses being classic dispersive behaviour. A corollary of this association is that, while losses are still increasing with frequency, inductance must still be diminishing. Hence internal inductance, although very small at high frequencies, never vanishes completely.

Internal inductance can usually be included in mathematical models by the simple expedient of adding it to the inductance obtained from external fields. Whether it is worthwhile to do so depends on the required accuracy; the inductance of small-signal RF coils, for example, being affected at roughly the 1% level at low MHz frequencies. It can be said however, that the issue of whether or not internal inductance is important can be settled without prevarication by putting it into the model as a matter of policy (especially if the model is part of a general-purpose computer program). That, of course, presumes that the calculation is straightforward.

The exact expression for the internal reactance per unit length of a round wire is obtained from the imaginary part of equation (3.2), i.e.;

$$\frac{2\pi f L_i}{\ell} = \frac{R_s}{\sqrt{2}\pi r} \left[ \frac{\text{Ber}(q)\text{Ber}'(q) + \text{Bei}(q)\text{Bei}'(q)}{\text{Ber}'(q)^2 + \text{Bei}'(q)^2} \right] \quad [\Omega / \text{m}] \quad (16.1)$$

where  $q = \frac{\sqrt{2}r}{\delta_i}$  and  $R_s = \sqrt{\pi f \mu \rho}$

To make the working to follow less cumbersome, let us define:

$$W(q) = \frac{\text{Ber}(q)\text{Ber}'(q) + \text{Bei}(q)\text{Bei}'(q)}{\text{Ber}'(q)^2 + \text{Bei}'(q)^2}$$

Hence, substituting for  $R_s$  in (16.1) and dividing throughout by  $2\pi f$ :

$$\frac{L_i}{\ell} = \frac{\sqrt{\pi f \mu \rho}}{2\sqrt{2}\pi^2 f r} W(q) \quad [\text{H} / \text{m}]$$

Removing a factor  $\mu/2\pi$  and recalling that  $\delta_i = \sqrt{\frac{\rho}{\pi f \mu}}$ , we get:

$$\frac{L_i}{\ell} = \frac{\mu}{2\pi} \frac{\delta_i}{r\sqrt{2}} W(q) = \frac{\mu}{2\pi} \frac{W(q)}{q} \quad [\text{H} / \text{m}] \quad \text{'exactly'} \quad (16.2)$$

Since the calculation of  $L_i$  uses the same Kelvin functions as are used for AC resistance (albeit in a different combination), its evaluation of for large values of  $q$  is subject to rounding error issues as before.

## 17. Limiting behaviour of internal inductance

At low frequencies, for a round wire:

$$\frac{L_i}{\ell} \rightarrow \frac{L_{i(dc)}}{\ell} = \frac{\mu}{2\pi} \frac{1}{4} \quad [ \text{H} / \text{m} ] \quad (17.1)$$

See, for example, ref.<sup>26</sup>. Note that, for non-magnetic materials, to an accuracy of about 1 part per thousand (0.1%),  $\mu = \mu_0 = 4\pi \times 10^{-7} \text{ H/m}$ . Hence, all non-magnetic round wires, regardless of diameter, have an internal inductance of 50 nH/m at low frequencies. This might not seem like much, but a small 10  $\mu\text{H}$  inductor wound on a low-permeability iron-powder toroid will require about 1 m of wire, and so the low-frequency internal inductance contribution will be about 0.5%. For a solenoid of similar inductance, it might take 2 or 3m of wire, and the LF internal inductance contribution will be 1% or more. Hence internal inductance is not important in rough inductance measurements, but it will cause modelling errors in more accurate work. Furthermore, for wires thinner than about 0.7 mm diameter, the internal inductance will change at its greatest rate in the MHz region.

At high frequencies:

$$\frac{L_i}{\ell} \rightarrow \frac{\mu}{2\pi} \frac{\delta_i}{d} = \frac{\mu}{2\pi} \frac{1}{q\sqrt{2}} \quad (17.2)$$

The limiting HF behaviour is for the internal inductance to become proportional to  $1/\sqrt{f}$ . Notice also, by comparison with (16.2):

$$W(\infty) = \frac{1}{\sqrt{2}}$$

## 18. Precise calculation of internal inductance (Flint, 0.01ppM)

The practical difficulties associated with using Kelvin Bessel functions are much the same as they are for AC resistance. At some point the rounding errors become intolerable and it is necessary to switch to an asymptotic form. The problem then, as in section 8, is that of determining a suitable bridging polynomial and choosing the branch point.

In order to compare different calculation methods in a manner independent of permeability, it is helpful to define a dimensionless internal inductance factor, i.e.:

$$\Theta = \frac{L_i}{L_{i(dc)}}$$

where  $\Theta$  (Greek "Theta") is a function of  $q$ , and  $L$  can be expressed either as inductance or as inductance per unit length. Hence, using (16.2) and (17.1):

$$\Theta(q) = \frac{W(q)/q}{1/4} = \frac{4W(q)}{q}$$

<sup>26</sup> **Fields and Waves in Communication Electronics**, 3rd Edition, Simon Ramo, John R. Whinnery, Theodore Van 4.5 (p182-186): Impedance of round wires.

and using (17.1) and (17.2):

$$\Theta_{\infty}(q) = \frac{4}{q\sqrt{2}} = \frac{2\sqrt{2}}{q}$$

The polynomial formula of Mingli and Yu (discussed in section 8) also gives rise to an asymptotic form for  $\Theta$ , i.e.;

$$\Theta_{MY} = \frac{4}{q} \frac{\text{Por}(q)}{[\text{Por}(q)^2 + \text{Poi}(q)^2]} \quad \pm < 0.1 \text{ ppM} \quad \text{for } q \geq 11.5$$

As before however, this formula requires a large number of empirical coefficients and, although extremely accurate, is not exactly contiguous with the Bessel calculation for finite  $q$ .

There is no analytical early-asymptotic form in this case, but a bridging polynomial can still be constructed. Here we will isolate the factor  $4/q$ , which is common to all of the calculation methods shown. Hence:

$$\Theta_{\infty}(q) = \frac{4}{q} \left[ \frac{1}{\sqrt{2}} + \frac{k_1}{q} + \frac{k_2}{q^2} + \frac{k_3}{q^3} \right]$$

It also happens that when  $k_1$  and higher coefficients are set to zero,  $\Theta_{\infty}$  comes out slightly larger than the exact calculation when  $q$  is finite. Hence it is possible to nudge the polynomial into coincidence with the Bessel calculation at some chosen point by using only negative coefficients. As before, constraining the coefficients (or, at least, being prepared to intervene manually if the signs alternate) prevents undulation about the zero-error line; so that the final overall error is the same as the rounding noise in the Bessel calculation just below the crossover point.

A crossover point at  $q=65$  was chosen, as for the AC resistance case. The noise in the Bessel calculation, using the same routine as for function Xirac, is  $\pm < 0.008$  ppM at this point. The coefficients were allowed to vary to fit the bridging polynomial to the Bessel calculation over the range from 56 to 65. The coefficient  $k_1$  was then adjusted manually to give near-exact contiguity at  $q=65$ , as estimated using a regression line on 201 closely-spaced points to smooth the Bessel calculation data. Finally, the determined polynomial was checked against the Mingli-Yu formula. The differences were much the same as for the AC resistance case (0.02ppM at  $q=65$  and 0.01ppM at  $q=83$ ), and as explained before, are due mainly to the error in the Mingli-Yu formula.

The bridging polynomial is used in the OO Basic macro function 'Flint', which is shown in the box below. The function name is coined as "Factor for internal inductance". The determined coefficients are as given in the program code<sup>27</sup>. The Kelvin function calculation routine is the same as for function Xirac (section 8), and so is not commented.

Flint requires the argument:

$$q = \frac{\sqrt{2}r}{\delta_i} = \frac{d}{\delta_i\sqrt{2}} \quad , \quad \text{where} \quad \delta_i = \sqrt{\frac{\rho}{\pi f \mu}} \quad .$$

and returns the value  $\Theta$  within 8 parts in  $10^9$ . Internal inductance per unit length is then:

<sup>27</sup> The calculations used in the development are given in the worksheet: **Flint.ods**, and the code can be copied using the macro editor as before.

$$\frac{L_i}{\ell} = \frac{\mu}{8\pi} \Theta \quad [ \text{H} / \text{m} ]$$

Flint is used for evaluation of the approximations discussed in the following sections; but of course, it can also be used directly for circuit modelling if execution speed is not an issue.

```

Function Flint(ByVal q as double) as double
'Calculates internal inductance factor within 0.01ppM.
Dim n%, a#, b#, c#, br#, bi#, dr#, di#, brtm#, bitm#, drtm#, ditm#
if q=0 then
  Flint=1
elseif q>65 then
'Use asymptotic polynomial
  Flint=(4/q)*(1/sqr(2)-0.00000088/q-0.264887/q^2-0.394747/q^3)
else
'Perform Bessel function calculation
  brtm=1
  bitm=q*q/4
  drtm=-q*q*q/16
  ditm=q/2
  br=1
  bi=bitm
  dr=drtm
  di=ditm
  n=0
  do
    n=n+1
    a=q/(4*n-2)
    b=q/(4*n)
    c=q/(4*n+2)
    brtm=-brtm*a*a*b*b
    bitm=-bitm*b*b*c*c
    drtm=-drtm*b*c*c*q/(4*n+4)
    ditm=-ditm*a*b*b*c
    br=br+brtm
    bi=bi+bitm
    dr=dr+drtm
    di=di+ditm
  loop until abs(brtm/br)<1E-12
  Flint=(4/q)*(br*dr+bi*di)/(dr*dr+di*di)
end if
end function

```



## 19. Asymptotically correct approximations

The most basic allowance for internal inductance in HF circuit modelling can be made by using the high-frequency limiting case as the formula. Thus:

$$\frac{L_i}{\ell} = \frac{\mu}{2\pi} \frac{\delta_i}{d} \quad [ \text{H} / \text{m} ] \quad \delta_i \ll d \quad \text{Thick conductor approximation (TCA)}$$

This expression however, has the defect that it goes to infinity as  $f \rightarrow 0$ , whereas it should go to  $\mu/8\pi$ . We can resolve this problem in a manner analogous to the approach used in section 9, by defining a modified 'skin-depth' parameter  $\delta_i''$  chosen so that  $\delta_i'' \rightarrow \delta_i$  at high frequencies, and  $\delta_i''/d \rightarrow 1/4$  at low frequencies. A suitable form for  $\delta_i''$  is:

$$\delta_i'' = \delta_i \left[ 1 - \exp\left(-\frac{a d}{\delta_i}\right) \right]$$

Hence:  $\delta_i'' \rightarrow a d$  as  $\delta_i \rightarrow \infty$ , and so  $a = 1/4$ . This gives:

$$\frac{L_i}{\ell} = \frac{\mu}{2\pi} \frac{\delta_i}{d} \left[ 1 - \exp\left(-\frac{d}{4\delta_i}\right) \right] \quad [ \text{H} / \text{m} ] \quad \text{1}^{\text{st}}\text{-order asymptotically correct approximation (ACA1)}$$

Notice that this approximation is not referred to as TED. This is because the modified 'skin depth' parameter has no direct physical significance in the internal inductance case. AC resistance is dependent only on the current density magnitude profile, and so can be related to a simple abrupt-cutoff (annular uniform conduction) model. Internal inductance however depends on both the magnitude and relative phase of the current density at any point within the conductor, and so cannot be trivially related to an equivalent region of uniform conduction just below the surface.

ACA1 can also be expressed as an internal inductance factor:

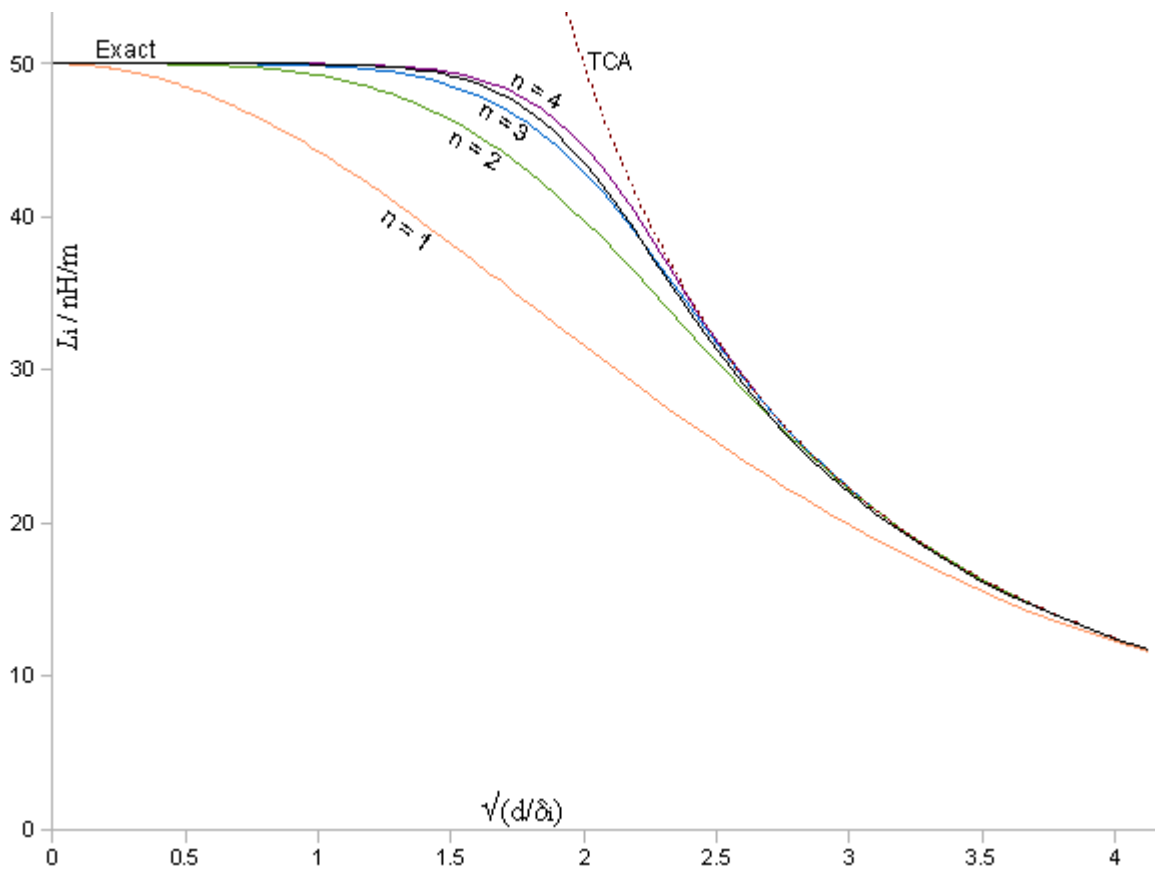
$$\Theta(q) = \frac{4}{q\sqrt{2}} \left[ 1 - \exp\left(-\frac{q\sqrt{2}}{4}\right) \right]$$

Further ACAs can now be generated in the manner discussed in section 10:

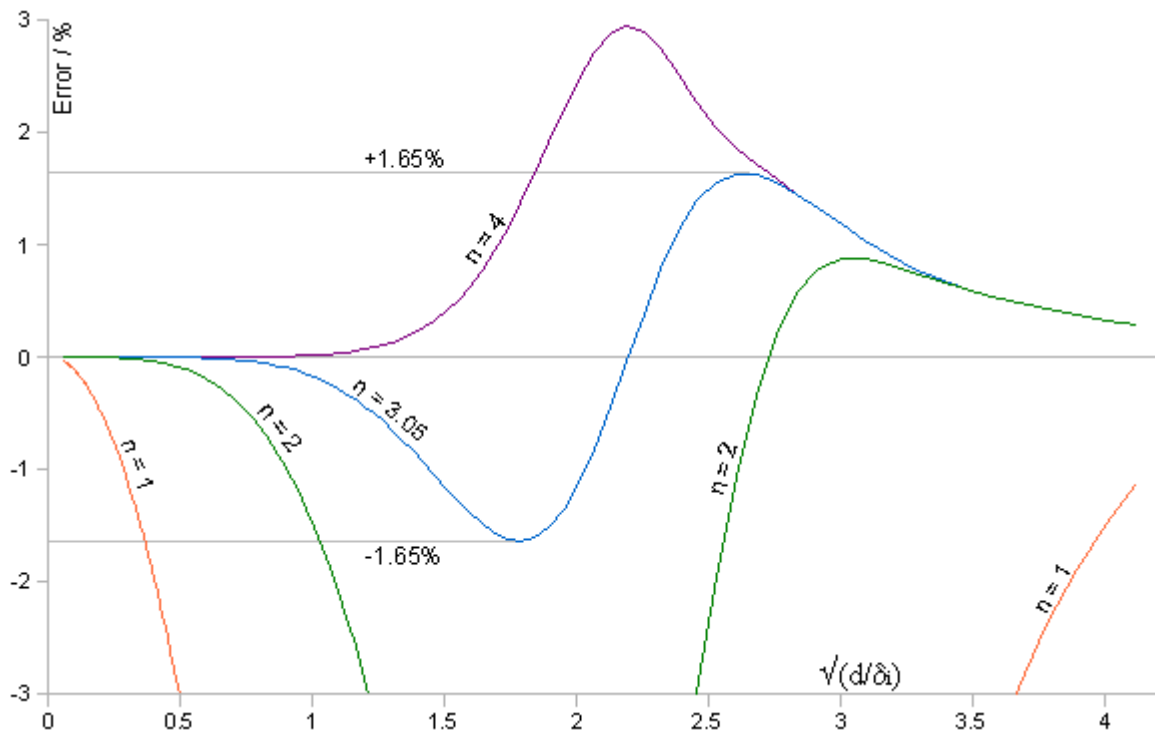
$$\frac{L_i}{\ell} = \frac{\mu}{2\pi} \frac{\delta_i}{d} \left[ 1 - \exp\left(-\left[\frac{d}{4\delta_i}\right]^n\right) \right]^{\frac{1}{n}} \quad [ \text{H} / \text{m} ] \quad \text{n}^{\text{th}}\text{-order asymptotically correct approximation (ACAn)}$$

$$\Theta(q) = \frac{4}{q\sqrt{2}} \left[ 1 - \exp\left(-\left[\frac{q\sqrt{2}}{4}\right]^n\right) \right]^{\frac{1}{n}}$$

The effect of varying  $n$  is shown in the graph below (spreadsheet calculation **Li\_aca.ods**, sheet 1). The 'exact' calculation is obtained by using the function **Flint(q)** described in section 18.



The  $n=1$  case is a rather poor approximation, and best promise is shown for  $n$  in the 3 to 4 region. Error curves are plotted below.



Values of  $n$  in the region of 3 produce S-shaped error curves, and  $n=3.05$  makes the positive and negative maximum error magnitudes the same. Hence a simple continuous formula for internal inductance accurate to  $\pm 1.65\%$  is:

**$L_i$ -ACA3.05 (1.65%)**

$$\frac{L_i}{\ell} = \frac{\mu}{2\pi} \frac{\delta_i}{d} \left[ 1 - \exp\left(-\left[\frac{d}{4\delta_i}\right]^{3.05}\right) \right]^{\frac{1}{3.05}} \quad [ \text{H} / \text{m} ] \quad \pm 1.65\% \quad \text{ACA3.05}$$

The corresponding internal inductance factor is:

$$\Theta(q) = \frac{4}{q\sqrt{2}} \left[ 1 - \exp\left(-\left[\frac{q\sqrt{2}}{4}\right]^{3.05}\right) \right]^{\frac{1}{3.05}} \quad \pm 1.65\%$$

where

$$q = \frac{d}{\delta_i\sqrt{2}} \quad \text{and} \quad \delta_i = \sqrt{\frac{\rho}{\pi f \mu}}$$

## 20. Multiple parameter ACAs

One of the features of the single-parameter ACAs discussed in the previous section is that they converge relatively slowly with the exact expression as the frequency increases. This is because there is no early asymptotic form on which to base the approximation. In section 18, in constructing a function for precise calculation, the problem was circumvented by using a bridging polynomial. That technique can bring the asymptotic form arbitrarily close to the exact value at any chosen point. Moreover, as will now be demonstrated; the TED method of section 9 can be used to turn any singly-asymptotic form into a doubly asymptotic form. This allows us to produce multi-parameter ACAs that offer increased latitude in adjusting the shape and runout of the error function.

Recall from section 9 that:

$$x' = x \left[ 1 - \exp\left(-\frac{a}{x}\right) \right] \rightarrow a \quad \text{as } x \rightarrow \infty$$

Therefore:

$$\Theta_{da}(q) = \frac{b}{aq} \left[ 1 - \exp\left(-\frac{aq}{b}\right) \right] \rightarrow 1 \quad \text{as } q \rightarrow 0$$

This means that if we have a singly-asymptotic approximation of the form:

$$\Theta_a(q) = \frac{b}{aq}$$

It can be turned into a doubly-asymptotic approximation by the TED method. Furthermore, there is nothing to say that the coefficient  $b$  cannot be a function of  $q$ ; although there are some some restrictions on the choice of function. In this case we have:

$$0 < q < \infty$$

i.e.,  $q$  is always finite and positive.

We can start by considering a candidate function of the form:

$$\Theta_a'(q) = \frac{1}{aq} \left[ 1 + \frac{k_1'}{q} \right]$$

so that

$$b = 1 + \frac{k_1'}{q}$$

This expression however, while well behaved when  $q \rightarrow \infty$ , explodes when  $q \rightarrow 0$  and can take on a value of zero at some point if  $k_1'$  is negative (which it will be for the present purpose). We can however ensure that  $b$  will always be finite and positive for  $k_1' > -1$  by replacing  $q$  with  $q+1$ , i.e.:

$$b = 1 + \frac{k_1}{q+1}$$

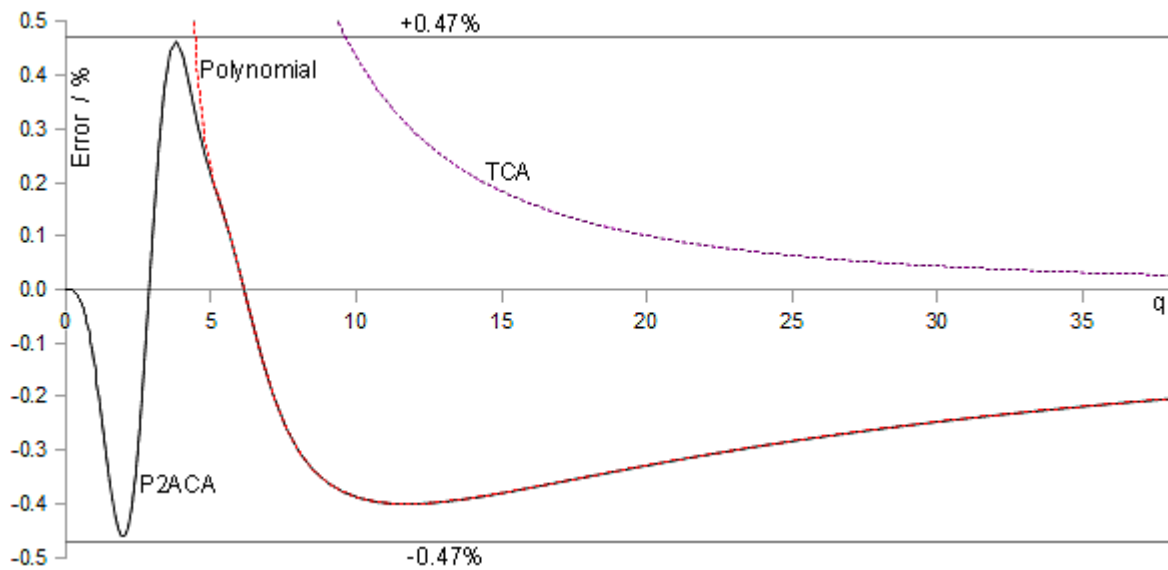
This change makes very little difference when  $q$  is large, but ensures that the resulting ACA is always continuous when  $q$  is small. So now, we have a modified polynomial as a starting approximation, and it should be obvious that this can be extended to higher order as necessary, i.e.:

$$\Theta_a(q) = \frac{1}{aq} \left[ 1 + \frac{k_1}{q+1} + \frac{k_2}{q^2+1} + \dots \right] = \frac{b}{aq}$$

Finally, we can adjust the order when converting this into a doubly asymptotic form:

$$\Theta_{\text{dan}}(q) = \frac{b}{aq} \left[ 1 - \exp\left(-\left[\frac{aq}{b}\right]^n\right)\right]^{\frac{1}{n}}$$

An ACA with two adjustable parameters (P2ACA) was used to match the exact calculation as accurately as possible and resulted in an approximation good to  $\pm 0.47\%$ . The error graph is shown below (spreadsheet: **Li\_aca.ods**, sheet 2)



The formula is:

### Li-P2ACA (0.47%)

$$\Theta_{\text{dan}}(q) = \frac{4b}{q\sqrt{2}} \left[ 1 - \exp\left(-\left[\frac{q\sqrt{2}}{4b}\right]^{3.563}\right)\right]^{\frac{1}{3.563}} \quad \text{where} \quad b = 1 - \frac{0.09}{q+1} ,$$

$$q = \frac{d}{\delta_i \sqrt{2}} \quad \text{and} \quad \delta_i = \sqrt{\frac{\rho}{\pi f \mu}} .$$

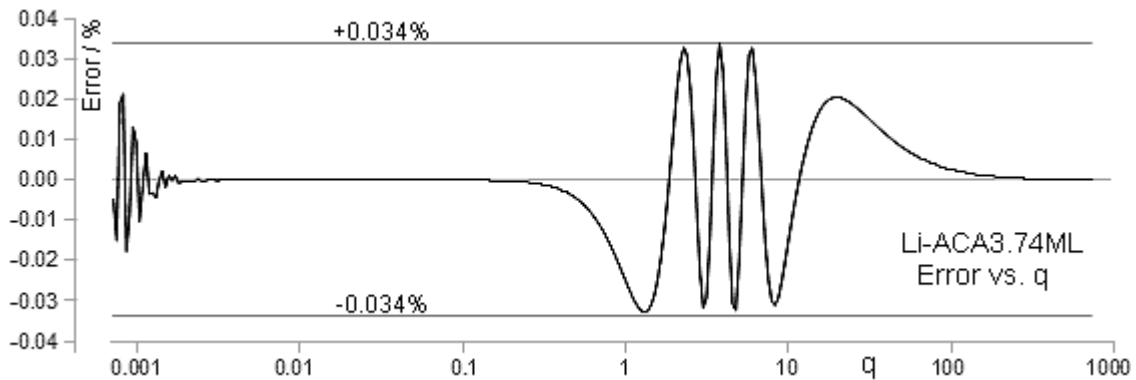
Internal inductance per unit length is given by:  $\frac{L_i}{\ell} = \frac{\mu}{8\pi} \Theta(q) .$

## 21. Modified Lorentzian correction

For ML correction, an error function that is all-positive or all-negative is required. Adopting the variable order (single parameter) ACA of section 19; it can be seen from the graphs of that section that solutions will not be possible in the region of  $n=3$ , where the error-curve is s-shaped, but we might reasonably expect solutions in the vicinities of  $n=1$  and  $n=4$ .

A machine-optimised formula accurate to 0.2% (with  $n=1.22$ ) was found by starting with a manual optimisation at  $n=1$  and then allowing the Solver program to find the best parameters. Starting with  $n=4$  however, it was possible to get the error down to 0.12% with manual optimisation, and that error reduced to 0.07% with  $n$  shifted to 3.8. Subsequent machine optimisation ended up with  $n=3.74$  and a maximum runout error of  $\pm 0.034\%$ . Divisor and multiplier ML corrections were tried (as for every case studied) and the two approaches produced almost identical error curves (albeit with slightly different parameters). With little basis on which to choose between the two, the parameters of the divisor corrected formula were truncated to 5 or less decimal places and subjected to final hand adjustment to re-minimise the runout.

The resulting formula (ACA3.74ML) provides very good accuracy for practical purposes. The variable-order ACA approach however involves exponentiation followed by reverse-exponentiation; and when  $n$  is relatively large, this can lead to floating-point rounding error when  $q$  is small. The issue is demonstrated in the error graph below, where it can be seen that the rounding error is becoming significant when  $q < 0.003$ . To put the problem into perspective however, the rounding error dies out at a frequency of about 1 Hz for copper wires of 0.2 mm diameter, and 10 Hz for wires of 0.05 mm diameter. Hence it is not likely to cause difficulties in RF modelling applications. Furthermore, if a routine that calculates the internal inductance factor returns a value of 1 for  $q < 0.01$  (as is done by the Basic macro function given below) then the error at the crossover is less than 0.1 ppM.



Fitting calculations are given in the spreadsheet: **Li\_acagt3ml.ods**. The formula is:

### **Li-ACA3.74ML (0.034%)**

$$\frac{L_i}{\ell} = \frac{\mu}{2\pi} \frac{\delta_i}{d(1+y)} \left[ 1 - \exp\left(-\left[\frac{d}{4\delta_i}\right]^{3.74}\right) \right]^{\frac{1}{3.74}} \quad [\text{H/m}] \quad \pm 0.034\% \quad \text{ACA3.74ML}$$

$$y = \frac{0.02369}{\left[1 + 0.2824 \left(z^{1.4754} - z^{-2.793}\right)^2\right]^{0.8955}}, \quad z = 0.27445q, \quad q = \frac{d}{\delta_i \sqrt{2}}, \quad \delta_i = \sqrt{\frac{\rho}{\pi f \mu}}$$

Expressed as an internal inductance factor, with variables defined as above, this becomes:

$$\Theta(q) = \frac{4}{q\sqrt{2}(1+y)} \left[ 1 - \exp\left(-\left[\frac{d}{4\delta_i}\right]^{3.74}\right) \right]^{\frac{1}{3.74}} \pm 0.034\%$$

so that  $\frac{L_i}{\ell} = \frac{\mu}{8\pi} \Theta(q)$  [ H / m ].

A Basic macro that returns  $\Theta$  using this formula is given in the box below:

```
Function FL3A74ML(ByVal q as double) as double
'Calculates internal inductance factor within 0.034%.
Dim z as double, y as double
if q<0.01 then
  FL3a74ml=1
else
  FL3a74ml = (4/q)/sqr(2)
  FL3a74ml = FL3a74ml*(1-exp(-1*(1/FL3a74ml)^3.74))^(1/3.74)
  z = 0.27445*q
  y = 0.02369/(1+0.2824*(z^1.4754-z^-2.793)^2)^0.8955
  FL3a74ml = FL3a74ml/(1+y)
end if
end function
```

## 22. Polynomial ACA with ML correction

What follows in this section is more a demonstration of what can be done (rather than what has to be done) when using the ACA-ML fitting approach. In section 20, the number of parameters in the initial singly-asymptotic approximation was increased by incorporating a polynomial. The procedure for converting the starting approximation ( $\Theta_\infty$  say) into a doubly-asymptotic form is then:

$$\Theta_{da} = \Theta_\infty \left[ 1 - \exp\left(-\Theta_\infty^{-n}\right) \right]^{\frac{1}{n}}$$

where the principal restriction on  $\Theta_\infty$  is that it must remain finite for any input argument within the allowed range. For the internal inductance problem, a suitable form for the singly-asymptotic polynomial is:

$$\Theta_\infty(q) = \frac{4}{q\sqrt{2}} \left[ 1 + \frac{k_1}{q+1} + \frac{k_2}{q^2+1} + \frac{k_3}{q^3+1} + \frac{k_4}{q^4+1} + \dots \right]$$

Notice that this is capable of returning a value of zero for finite  $q$  if there is an unfavourable choice of empirical coefficients ( $k_1, k_2$ , etc.), but such a situation will not occur if the correction provided by adjusting the coefficients is reasonably small. Also notice that, when the coefficients are all set

to zero,  $\Theta_\infty$  is simply the thick-conductor approximation for the internal inductance factor.

So now, starting with the thick conductor approximation, we can apply ML correction to  $\Theta_{da}$ . The machine fitting process will crush the runout error into a series of undulations. The polynomial however is also capable of producing undulations about the zero-error line; and so, if its coefficients are subsequently allowed to vary, there will be a tendency for it to cancel the undulations left over after ML correction. Furthermore, if the ML parameters, the order (n), and the polynomial coefficients are all allowed to vary simultaneously, there will be a repartitioning of errors between the various stages of correction (presuming that no discontinuities are encountered).

This fitting exercise was undertaken, partly out of curiosity, to see if an ACA-ML solution could be found with  $n < 3$ ; i.e., without the rounding error that occurs for small q with the ACA3.74ML formula. Starting with  $n=1$  and all polynomial k-coefficients set to zero, the best ML fit (ACA1ML-divisor) gave a maximum runout of 0.28%. Allowing n to vary reduced this to 0.2% with  $n=1.222$ . This is not particularly impressive; but a considerable improvement occurred when the polynomial coefficients were introduced. With  $k_1$  allowed to vary, n drifted up to 1.416 and the runout fell to 0.066%. With both  $k_1$  and  $k_2$  allowed to vary, n changed to 1.387 and the runout fell to 0.055%. With  $k_1$ ,  $k_2$  and  $k_3$  allowed to vary, n changed to 1.684 and the runout fell to 0.023%. Inclusion of  $k_4$  did not give any improvement, the reason being that terms in  $q^{-4}$  die-out too rapidly to have a significant effect in the region where the undulations lie. Finally, the determined parameters were transferred to the ML-multiplier formula (except for h, which was adjusted by hand) to be used as the starting point for a fit with all allowed to vary. The multiplier correction proved to be better than the divisor, giving a maximum runout of 151 ppM, i.e., a reasonably claimable accuracy of  $\pm 0.016\%$ . The order n in this case came out at 1.582; and in a test simulation for a 0.1 mm diameter copper wire, it was found that the rounding error was considerably less than 1 ppM for frequencies as low as  $10^{-8}$  Hz ( $\sim 0.3$  cycles per year), which is probably adequate for most purposes. The formula is shown below<sup>28</sup>.

### Li-PACAML (0.016%)

$$\Theta_\infty = \frac{4}{q} \frac{1}{\sqrt{2}} \left[ 1 + \frac{0.01209}{(q+1)} - \frac{0.63523}{(q^2+1)} + \frac{0.16476}{(q^3+1)} \right] \quad \text{where} \quad q = \frac{d}{\delta_i \sqrt{2}} \quad \text{and} \quad \delta_i = \sqrt{\frac{\rho}{\pi f \mu}} \quad ,$$

$$\Theta_{da} = \Theta_\infty \left[ 1 - \exp\left(-\Theta_\infty^{-1.5819}\right) \right]^{\frac{1}{1.5819}} \quad \text{and}$$

$$\Theta = \Theta_{da} (1 - y) \quad \text{where}$$

$$y = \frac{-0.198584}{\left[ 1 + 0.25741 (z^{1.2652} - z^{-0.39709})^2 \right]^{2.62343}} \quad \text{and} \quad z = 0.38691q \quad .$$

$$L_i = \ell \frac{\mu_{(i)}}{8\pi} \Theta \quad [ \text{Henrys} ] \quad .$$

Admittedly, this formula has a rather large number of empirical parameters, but then again, it does calculate internal inductance to within 151 ppM with not a Bessel function in sight. An OO Basic macro version is given below.

<sup>28</sup> The fitting calculation is given in the spreadsheet: **Li\_pacaml.ods** (sheet 2).



```

Function Flpml(ByVal q as double) as double
'Calculates internal inductance factor within 151 ppM.
'D. W. Knight, version 1.01, 2016-01-25.
  Dim y as double, z as double, zz as double
  if q<0.0001 then
    Flpml=1
  else
    Flpml=(4/q)*(1/sqr(2))*(1+0.01209/(q+1)-0.63523/(q*q+1)+0.16476/(q*q*q+1))
    Flpml=Flpml*(1-exp(-1*Flpml^-1.5819))^0.63215121
    z=0.38691*q
    zz=Z^1.2652-Z^-0.39709
    y=-0.198584/(1+0.25741*ZZ*ZZ)^2.62343
    Flpml=Flpml*(1-y)
  end if
end function

```

The function 'Lintern()' shown below performs the complete internal inductance calculation for a length of round wire, calling the function 'Flpml()' for the internal inductance factor. Lintern requires the wire length and diameter, the frequency, the relative permeability of the wire (1 for Cu or Ag), and a quantity  $K_{iacs}$ .  $K_{iacs}$  is the wire conductivity as a proportion of the International Annealed Copper Standard<sup>29 30</sup>. For electrical copper wire at 20°C,  $K_{iacs} = 1$ . Note that at 20°C, the resistivity  $\rho$  of IACS standard copper is  $17.241 \times 10^{-9} \Omega\text{m}$ , whereas (for example) the resistivity of silver at this temperature is  $15.87 \times 10^{-9} \Omega\text{m}$ . Therefore, for silver wire (bearing in mind that conductivity is the reciprocal of resistivity):

$$K_{iacs} = 17.241 / 15.87 = 1.086$$

i.e., at 20°C, silver is 1.086 times more conductive than IACS copper.

```

Function Lintern(Byval lw as double, d as double, f as double, Kiacs as double, mur as double)
'Internal inductance of a round wire. D W Knight, version 1.00, 2016-01-09.
'Calls function Flpml(q) for inductance factor Theta.
' lw = length of wire / m
' d = wire diameter / m
' f = frequency / Hz
' Kiacs is proportionate conductivity relative to IACS. For electrical Cu at 20 deg. C, Kiacs = 1
' mur = relative permeability of the wire = 1 for non magnetic conductors
Dim rho as double, delta as double, q as double, Theta as double
rho = 17.241E-9 / Kiacs
delta = sqr( rho / ( 4E-7*pi()*pi() * f * mur ) )
q = d / ( delta * sqr(2) )
Theta = Flpml(q)
Lintern = lw * 0.5E-7 * mur * Theta
end function

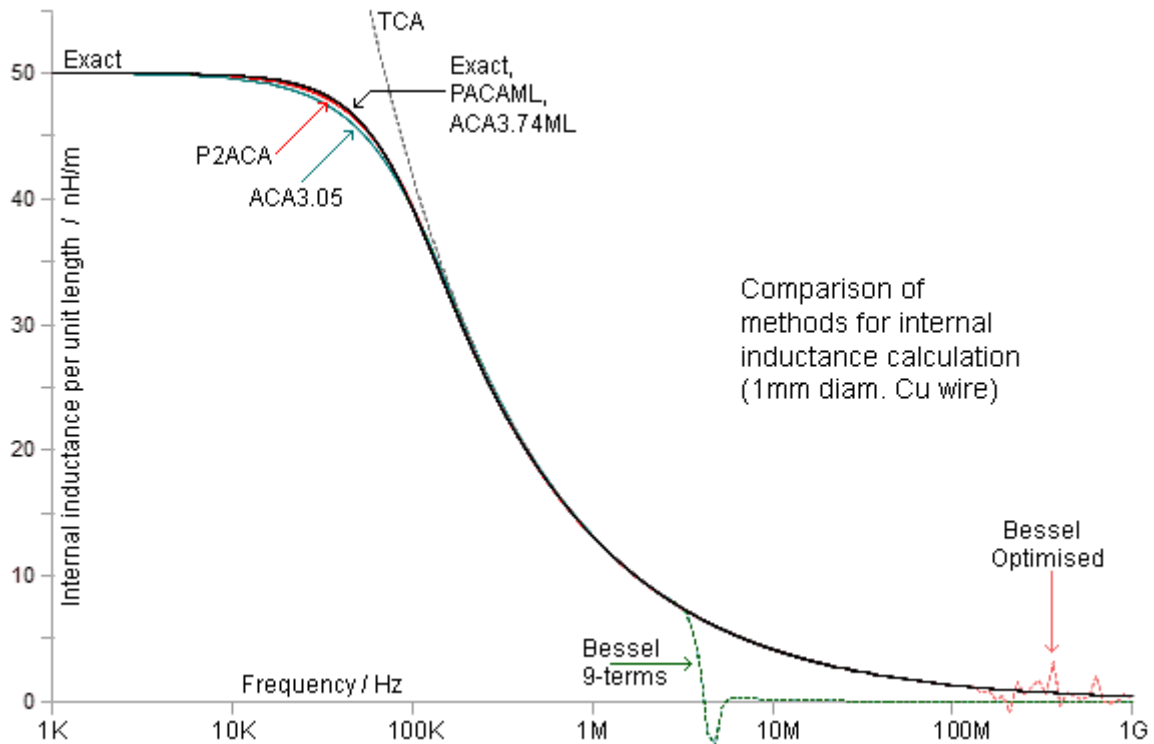
```

29 **Copper wire tables.** Bureau of Standards Circular No. 31. 3rd edition. 1914. International Annealed Copper Standard. pages 8 - 13. Available from <http://g3ynh.info/zdocs/comps/>

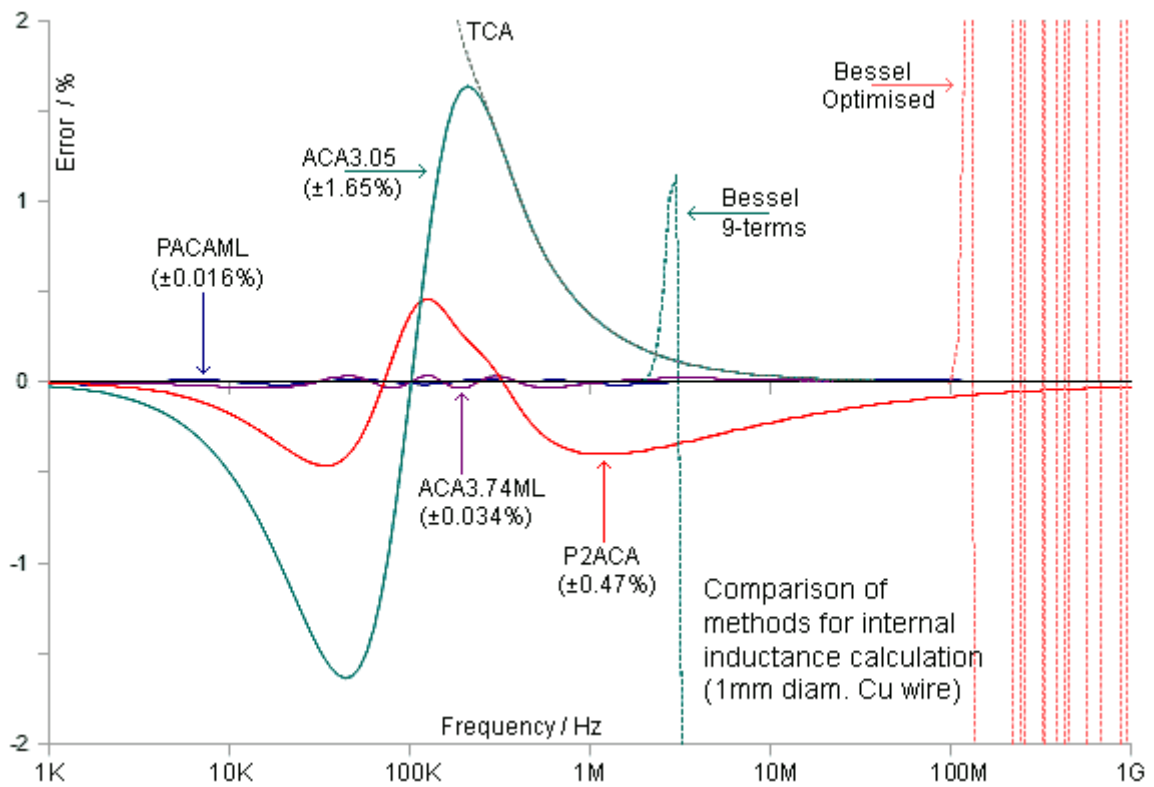
30 **Coppers for electrical purposes.** V A Callcut. Proc. IEEE, Vol. 133, Pt. A, No. 4. June 1986.

### 23. Comparison of calculation methods

Example calculations using some of the methods discussed in the previous sections are shown plotted below (see spreadsheet: **Zint\_calcs.ods**, sheet 2). The simulation is for a 1 mm diameter non-magnetic wire, for which the main part of the dispersion occurs in the 10 kHz to 1 MHz region. 1 mm however is quite a large wire size for small-signal RF inductors and transformers. Reducing the diameter moves the dispersion to higher frequency without otherwise altering the appearance of the graph (i.e., it changes the frequency scale but does nothing else).



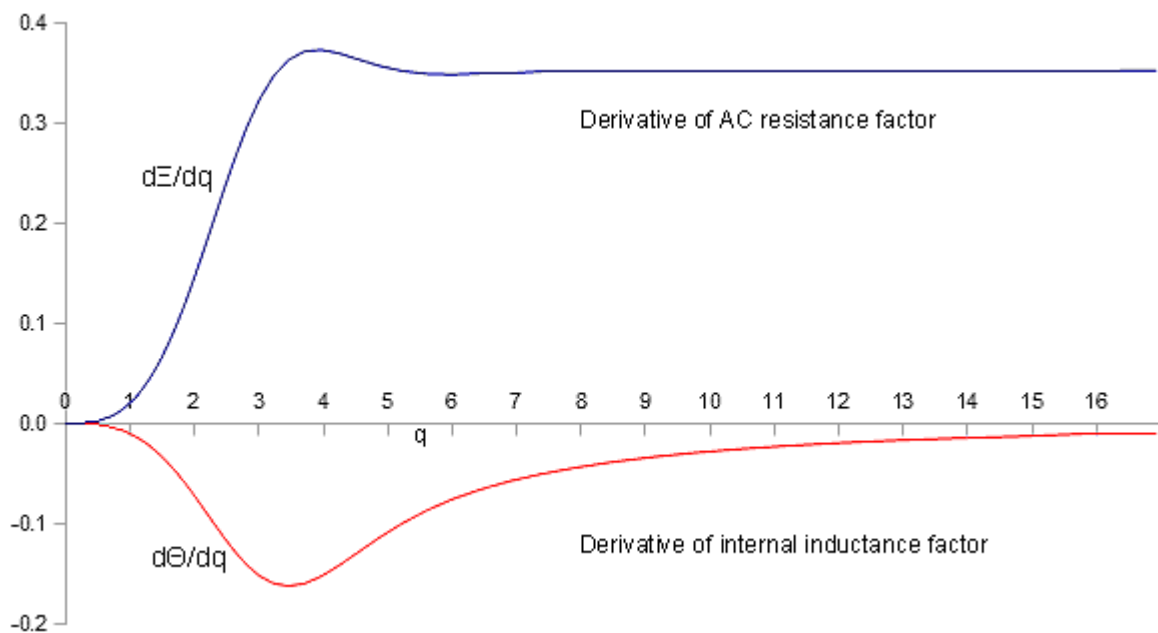
The error in the various methods is plotted below.



## 24. Concluding remarks

### Dimensional resonance:

The electromagnetic dispersion behaviour of an object having cylindrical symmetry is described by combinations of Bessel functions. The Bessel functions are infinite-order polynomials of an aperiodic oscillatory nature; and if a combination of such functions should happen to give a curve of overall smooth appearance, it is only because the combination has reduced or disguised the serpentine nature of its constituents. Thus, whenever we try to approximate a composite cylinder function using a smooth curve of relatively low order; we find, in the differences, evidence of the dispersion function's ability to veer suddenly in a way that no simple expression can match. This behaviour can be seen most easily for the internal impedance case by plotting graphs of the derivatives of the AC resistance and internal inductance factors ( $d\Xi/dq$  and  $d\Theta/dq$ ), as shown below<sup>31</sup>. The curves are oddly reminiscent of resonance-related dispersive behaviour; except that the frequency scale and the roles of the real and imaginary parts are reversed.



Note incidentally, that the graph abscissa  $q$  is proportional to the square root of frequency, i.e.:

$$q = \frac{\sqrt{2}r}{\delta_i} = d \sqrt{\frac{\pi f \mu}{2\rho}}$$

Classic dispersion relations arise from the way in which the polarisability of a medium varies as the frequency is swept across a natural resonance. On the low-frequency side of its resonant frequency, a natural oscillator can keep up with the field fluctuations of the incident radiation and can therefore store energy by virtue of its displacement from equilibrium against restorative forces. At the resonant frequency, the peak displacement from equilibrium is at its maximum, but then so too is the coupling to natural damping processes, and so the real part of the dispersion relation (the energy loss) comes to a peak. On the high frequency side, the oscillator can no longer keep up with the radiation, and the polarisability declines rapidly. The overall result is a stepwise decline in the imaginary part of the relationship (the refractive index) on crossing a resonance (sometimes with overshoot, depending on the rate of change), and a peak in the real part (the absorption spectrum) centred on the point where the rate of change of reactance is at its greatest. Thus the derivative of

<sup>31</sup> Worksheet: [Zint\\_calcs.ods](#), sheet 3

the real part appears in the description of the imaginary part. Natural and engineered materials have many resonances, and the peaks of energy absorption correspond to atomic and molecular spectra as well as to the gross internal-reflection resonances familiar to electrical engineers. The decline in refractive index as frequency increases (in steps as each resonance is put to the low-frequency side of the excitation) corresponds to the way in which EM radiation becomes less prone to scattering as the wavelength is reduced.

The characteristic signature of a natural resonance process is the Lorentzian line-shape function. The pure Lorentzian however, is only strictly characteristic of a special case encountered in the field of high-resolution spectroscopy, i.e., when the Q of the resonance is enormous and the peak so narrow that the function of frequency against which it is plotted is irrelevant. There is a more general form that applies to all electromagnetic resonances; a modified Lorentzian, asymmetric about the peak unless plotted on a logarithmic frequency scale, and known as the electrical resonance curve. The latter is the basis of the ML correcting function used in this work.

As a dispersive phenomenon, the frequency dependence of internal impedance appears unusual in that it is not associated with a natural resonance in any obvious sense. If we understand a dispersion region to be a range of frequencies over which an impedance function can undergo sudden change however, then we see what looks like resonant behaviour in second order; i.e., when some physically-related smooth function is subtracted to reveal the underlying waywardness. The physics moreover is perfectly clear: it is not a conventional resonance, but it is an optical interference process, i.e., the dispersion region is centred on the point where the depth of field penetration interacts most strongly with the thickness of the conductor. Indeed, the dispersion can be considered as a type of *dimensional resonance*, somewhat analogous to the the impedance effects seen in magnetically-cored inductors and transformers due to the standing waves allowed by the extremely low phase velocity for EM waves propagating in some types of core material<sup>32</sup>. As illustrated by Harold Wheeler<sup>33</sup>, the internal impedance problem can be envisaged in terms of a slow-wave propagation process normal to the conductor surface.

This study was prompted by the desire to produce an accurate model for a reference inductor with widely-spaced turns. Initial attempts at correcting data for internal impedance were made using thick-conductor approximations; with the early-asymptotic form of section 6 being preferred over the traditional version because the neglect of curvature is clearly deleterious. TCAs were soon found to be inconvenient however, because they restrict the experimental frequency range, and so the truncated exponential decay (TED) approach was conceived as a way of correcting the low-frequency behaviour. It was then found that the difference between TED and the exact AC resistance calculation (using Bessel functions) was a bell-shaped curve, similar to the electrical resonance curve, with principal parameters related to mathematical constants. At the time, it seemed that there might be an alternative derivation for the transition from low-frequency to high-frequency conduction behaviour, based on the idea of optical interference giving rise to a pseudo-resonance; but since it was not possible to obtain an exact fit between the two approaches, the idea was put aside. After that, attention was shifted to the use of ad-hoc adjustments to the ML function in order to obtain the best possible fit to accepted theory, and the outcomes are as reported. Still, it has to be said that the extracted ML function does have some physical significance, in that it characterises a transitional behaviour with a known cause; and if we sweep aside the pragmatic business of producing working formulae, it would be difficult to demonstrate an experimental difference between TED-ML and Heaviside's solution of Maxwell's equations. No superiority is claimed, but it does hint at the basis for a second-order analytical approach to some types of problem for which no exact solution of Maxwell's equations exists.

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32 "Soft Ferrites, Properties and Applications", E C Snelling. Butterworths. 2nd edition 1988. Dimensional resonance is discussed in sections 2.2.4.3(p36-37) and 4.2.3 (p146)

33 "Formulas for Skin Effect". H A Wheeler. Proc. IRE, Vol. 30, No. 9, Sept. 1942, p412-424. See Fig 3 and related discussion.

We end up with some useful approximation formulae, some insight of the skin effect, and a set of mathematical tools for the generation and correction of doubly-asymptotic approximations. The latter might prove to be of value in deriving first and second order models of otherwise analytically intractable phenomena, such as the high-frequency dispersive behaviour of helical inductors.

### **Proximity effect:**

The calculation methods discussed in this article are only strictly applicable to isolated conductors. Such conductors are realised in practice in go-and-return circuits for which the return path is a long way away from the segment of wire under consideration. When the conductor segment is part of a transmission-line, or part of an inductor, the assumption that it is negligibly affected by the magnetic field from other parts of the circuit is no longer valid. The practical consequence is known as the *proximity effect*, i.e., a current redistribution causing deviation from cylindrical symmetry, which gives rise to further frequency-dependent losses and variations of internal inductance. The current redistribution also alters the physical location of the equivalent filamentary conductor, and so affects external inductance. For the reference inductor mentioned above, the proximity effect was minimised by using a very large pitch to wire-diameter ratio ( $p/d > 10$ ).

### **Tubes and plated conductors:**

This is a study of solid homogeneous conductors. Expressions for the internal impedance of circular tubes are given by McLachlan<sup>34</sup> and Dwight<sup>35, 36</sup>. Expressions for coated conductors and thin-walled tubes are given in the 1<sup>st</sup> edition of Ramo, Whinnery & Van Duzer<sup>37</sup>, but have been omitted from the 3<sup>rd</sup> edition.

### **General:**

For an overview and bibliography of the subject of internal inductance and partial inductance modelling, see the review article by Holloway et al.<sup>38</sup>.

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34 **Bessel Functions for Engineers**. N W McLachlan. 2nd edition. Oxford, Clarendon Press 1955. Section 8.30 (p160-162) Skin effect in circular tubes.

35 Skin effect in tubular and flat conductors. H B Dwight. Trans. AIEE Vol. XXXVII July 1918. p1399-1403.

36 Skin effect and proximity effect in tubular conductors. H B Dwight. Trans. AIEE, Vol. XL1, Jan.1922. p189-198.

37 **Fields and Waves in Communication Electronics**, Simon Ramo, John R. Whinnery, Theodore Van Duzer, Publ. John Wiley & Sons Inc. 1965. Library of congress cat. card no. 65-19477.

5.19 (p298-301): Impedance of a coated conductor. 5.20 (p301-303): Impedance of a thin-walled tubular conductor.

38 **Partial and internal inductance: Two of Clayton R Paul's many passions**. C L Holloway, E F Kuester, A E Ruehli, G Antonini. IEEE Trans on EM compatibility. 55(4), Aug. 2013. p600- 613. <https://www.researchgate.net/publication/256670447>

## Notes

### Program code

Verified code listings are given in the text and provide insurance in the event of typographical error. The code will be correct in the event of disagreement with the formula.

Basic macros for calculating the AC resistance factor  $\Xi$  and the internal inductance factor  $\Theta$  are given in the Apache Open Office spreadsheet file **Zint\_calcs.ods**<sup>39</sup>. Note that any worksheet that calls an added-in function must not have the name of that function as the sole contents of any cell, the reason being that the spreadsheet will search for the name in the data before it searches the macro library (see, in the accompanying spreadsheets, that when a function name is put at the head of a column, it is placed in quotes). The macros can be viewed and copied using the Open Office "Tools > Macros > Organise Macros" menu.

### Revision history

2009-12-19: version 1.00: Hand optimised approximation formulae. Xirac 16 ppM.  $R_{ac}$  0.16%, Flint 9 ppM.  $L_i$  0,07%.

2010-03-09: version 2.00: Machine optimised approximation formulae.  $R_{ac}$  0.09%,  $L_i$  0.016%. Section 14 added. Xirac 0.01 ppM. Flint 0.01 ppM.

2016-04-06: v 2.07: Minor corrections and layout changes since v 2.00.

2016-04-27: v 2.08: Equations typeset in OO Math. Some errors found and corrected. Function 'Lintern' added in section 22 and Flpml updated to v 1.01.




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<sup>39</sup> available from <http://g3ynh.info/zdocs/comps/zint/index.html>

Also available from ResearchGate: <https://researchgate.net/publication/301674351>